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Supersymmetric Boundary Conditions in $\mathcal{N} = 4$ Super Yang-Mills Theory

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Abstract We study boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory that preserve one-half the supersymmetry. The obvious Dirichlet boundary conditions can be modified to allow some of the scalar fields to have a "pole" at the boundary. The obvious Neumann boundary conditions can be modified by coupling to additional fields supported at the boundary. The obvious boundary conditions associated with orientifolds can also be generalized. In preparation for a separate study of how electric-magnetic duality acts on these boundary conditions, we explore moduli spaces of solutions of Nahm's equations that appear in the presence of a boundary. Though our main interest is in boundary conditions that are Lorentz-invariant (to the extent possible in the presence of a boundary), we also explore non-Lorentz-invariant but half-BPS deformations of Neumann boundary conditions. We make preliminary comments on the action of electric-magnetic duality, deferring a more serious study to a later paper.

Keywords Quantum field theory · Boundary conditions

1 Introduction

Supersymmetric boundary conditions in two-dimensional supersymmetric sigma models have been much studied, because of their role in string theory and their importance in understanding mirror symmetry. There has been comparatively very little study of supersymmetric boundary conditions in four-dimensional supersymmetric gauge theories. In this paper, we begin such a study, focusing on the case of boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory that preserve one-half of the supersymmetry.

One obvious choice comes from Neumann boundary conditions for gauge fields, suitably extended to the rest of the supermultiplet; another obvious choice comes from Dirichlet boundary conditions for gauge fields. A hybrid of the two can be constructed using an

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involution (a symmetry of order two) of the gauge group. Special cases of these boundary conditions arise in string theory from D3-NS5 systems, D3-D5 systems, and D3-branes interacting with an orientifold five-plane. All three of these constructions have significant generalizations, as we explain in Sect. 2 of this paper, where we attempt a systematic survey of superconformal boundary conditions that preserve one-half of the supersymmetry.

Dirichlet boundary conditions, and its cousins, lead to an unusual phenomenon. A supersymmetric vacuum of $\mathcal{N}=4$ super Yang-Mills theory on a half-space is not uniquely determined by the boundary conditions and the values of the fields at infinity; even after this data is fixed, the theory has a moduli space of supersymmetric vacua that appear as solutions of Nahm's equations. This phenomenon is explored in Sect. 3.

The S-dual of this property of Dirichlet boundary conditions is that gauge theory with gauge group G and Neumann boundary conditions can be coupled to a boundary superconformal field theory with G symmetry. Here we make only a few preliminary remarks about S-duality, deferring a more serious study to a subsequent paper.

Though our main focus is on boundary conditions that preserve Lorentz invariance (to the extent that this is possible in the presence of a boundary) and even conformal symmetry, we also in Sect. 4 explore deformations of Neumann boundary conditions that preserve one-half of the supersymmetry but violate Lorentz invariance.

Because Nahm's equations play an important role in this paper, we mention a few references. These equations were originally introduced [1] to study solutions of the Bogomolny equation for monopoles. See [2] for a review in that context. They were originally related to D-branes in [3]. Subsequent D-brane work [4, 5] uncovered some of the issues involving D-branes, impurities, and discontinuities in Nahm's equations that will be relevant in Sect. 3. As we explain most fully in Sect. 2.6, the study of supersymmetric boundary conditions is closely related to the study of supersymmetric defects. Early references on supersymmetric defects via branes include [6–8].

A rough analog of our problem in statistical mechanics is to analyze Kramers-Wannier duality for the Ising model on a lattice of finite spatial extent. Kramers-Wannier duality exchanges order and disorder, so it exchanges ordered and disordered boundary conditions. The four-dimensional problem we study is somewhat similar. One of the main differences is that as the boundary is three-dimensional, the complexities of three-dimensional quantum field theory can enter in the analysis of boundary conditions.

2 Half-BPS Boundary Conditions

Our goal is to describe supersymmetric boundary conditions—and more generally supersymmetric domain walls—in four-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory. More specifically, we will describe boundary conditions that are maximally supersymmetric, which means that they preserve half of the full underlying supersymmetry and in fact half of the superconformal symmetry. The full superconformal symmetry of $\mathcal{N}=4$ super Yang-Mills is $PSU(4|4)$ (or $PSU(4|2,2)$, to be more precise about the signature), and the unbroken subgroup will be $OSp(4|4)$.

$\mathcal{N}=4$ super Yang-Mills theory is conveniently obtained [9] by dimensional reduction from ten dimensions. We begin in $\mathbb{R}^{1,9}$, with metric g_{IJ}, $I, J = 0, \dots, 9$ of signature $-++\cdots+$. Gamma matrices Γ_I obey $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}$, and the supersymmetry generator is a Majorana-Weyl spinor ε, obeying $\overline{\Gamma}\varepsilon = \varepsilon$, where $\overline{\Gamma} = \Gamma_0\Gamma_1\cdots\Gamma_9$. The fields are a gauge field A_I and Majorana-Weyl fermion Ψ, also obeying $\overline{\Gamma}\Psi = \Psi$. Thus, ε and Ψ both transform in the **16** of $SO(1,9)$. The supersymmetric action is

$$I = \frac{1}{e^2} \int d^{10}x \, \mathrm{Tr}\left(\frac{1}{2} F_{IJ} F^{IJ} - i \overline{\Psi} \Gamma^I D_I \Psi\right). \tag{2.1}$$

The conserved supercurrent is

$$J^I = \frac{1}{2} \mathrm{Tr} \, \Gamma^{JK} F_{JK} \Gamma^I \Psi, \tag{2.2}$$

and the supersymmetry transformations are

$$\delta A_I = i \overline{\varepsilon} \Gamma_I \Psi, \tag{2.3}$$

$$\delta \Psi = \frac{1}{2} \Gamma^{IJ} F_{IJ} \varepsilon. \tag{2.4}$$

We reduce to four dimensions by simply declaring that the fields are allowed to depend only on the first four coordinates $x^0, \ldots, x^3$. This breaks the ten-dimensional Lorentz group $SO(1,9)$ to $SO(1,3) \times SO(6)_R$, where $SO(1,3)$ is the four-dimensional Lorentz group and $SO(6)_R$ is a group of R-symmetries. Actually, the fermions transform as spinors of $SO(6)_R$, and the R-symmetry group of the full theory is really $Spin(6)_R$, which is the same as $SU(4)_R$. The ten-dimensional gauge field splits as a four-dimensional gauge field A_μ, $\mu = 0, \ldots, 3$, and six scalars fields A_{3+i}, $i = 1, \ldots, 6$ that we rename as Φ_i. They transform in the fundamental representation of $SO(6)_R$. The supersymmetries ε and fermions Ψ transform under $SO(1,3) \times SO(6)_R$ as $(\mathbf{2}, \mathbf{1}, \mathbf{4}) \oplus (\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$, where $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ are the two complex conjugate spinor representations of $SO(1,3)$ and $\mathbf{4}, \overline{\mathbf{4}}$ are the two complex conjugate spinor representations of $SO(6)_R$.

Now we want to restrict to a half-space $x^3 \geq 0$ and introduce a supersymmetric boundary condition. We sometimes write y for x^3. We will consider (until Sect. 4) only boundary conditions that are invariant under $SO(1,2)$ Lorentz transformations that leave fixed the plane $y = 0$, and moreover, are also invariant under the larger group $SO(2,3)$ of conformal transformations that preserve this plane. It is impossible to also preserve the full R-symmetry group $SO(6)_R$, because, as we explain momentarily, invariance under $SO(1,2) \times SO(6)_R$ would imply invariance under all of the supersymmetries, or none. Preserving all supersymmetries would imply preserving all translation symmetries (since the commutator of two supersymmetries is a translation generator), and this is incompatible with having a boundary at $y = 0$.

The problem with $SO(1,2) \times SO(6)_R$ as a symmetry of a boundary condition is that under $SO(1,2)$, the two spinor representations of $SO(1,3)$ are equivalent and real, and so under $SO(1,2) \otimes SO(6)_R$, the supersymmetries transform as $\mathbf{2} \otimes (\mathbf{4} \oplus \overline{\mathbf{4}})$. Because the $\mathbf{4}$ and $\overline{\mathbf{4}}$ are inequivalent complex representations, it follows that the space of supersymmetries has no non-trivial invariant real subspace. To get such a subspace, we must reduce $SO(6)_R$ to a suitable subgroup.

Actually, in order for a boundary condition to be conformally invariant, the subgroup of $SO(6)_R$ must be $SO(3) \times SO(3)$, embedded in $SO(6)_R$ in the obvious way. Indeed, the superconformal group that contains the conformal group $SO(2,3)$ and has half of the full superconformal symmetry of $\mathcal{N} = 4$ super Yang-Mills theory[1] is $OSp(4|4)$, whose bosonic part

[1] We recall that this superconformal symmetry is $PSU(4|4)$, with 32 supercharges, half of which are preserved in $OSp(4|4)$.

is $SO(4) \times Sp(4,\mathbb{R})$. Recall that $Sp(4,\mathbb{R})$ is a double cover of $SO(2,3)$, and that $SO(4)$ is a double cover of $SO(3) \times SO(3)$. $SO(4)$ is the R-symmetry subgroup preserved by a boundary condition with $OSp(4|4)$ symmetry, and that is why a conformally invariant boundary condition must break $SO(6)_R$ to $SO(3) \times SO(3)$ or $SU(4)_R$ to $SO(4)$.

Under $SO(4)_R$, the **4** and $\overline{\mathbf{4}}$ of $SU(4)_R$ are real and equivalent, both transforming as $(\mathbf{2},\mathbf{2})$ under $SO(4)_R$, viewed as a double cover of $SU(2) \times SU(2)$. So we can take any linear combination $\mathbf{4}'$ of the **4** and $\overline{\mathbf{4}}$, and look for a boundary condition that preserves a subspace $\mathbf{2} \otimes \mathbf{4}'$ of the global supersymmetries. Our boundary conditions will also have manifest conformal invariance, which will ensure the full $OSp(4|4)$.

Although, up to isomorphism, the unbroken supergroup does not depend on which linear combination of the **4** and $\overline{\mathbf{4}}$ is chosen in this construction, the boundary conditions that we can construct in $\mathcal{N}=4$ super Yang-Mills theory do depend very much on this choice. That leads to much of the richness of the theory.

$PSU(4|4)$ has a one-parameter group of outer automorphisms that is responsible for the existence of a family of inequivalent embeddings of $OSp(4|4)$. Represent an element M of the superalgebra $PSU(4|4)$ by a supermatrix

$$M = \begin{pmatrix} S & T \\ U & V \end{pmatrix} \tag{2.5}$$

where S and V are bosonic 4×4 blocks and U and T are fermionic ones. M is unitary and unimodular (in the $\mathbb{Z}_2$-graded sense), and in $PSU(4|4)$, M is equivalent to λM for any scalar λ. Then $PSU(4|4)$ has a group $U(1)$ of outer automorphisms, acting by $M \to VMV^{-1}$ with

$$V = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta \in \mathbb{R}. \tag{2.6}$$

Conjugation by $U(1)$ generates the one-parameter family of embeddings of $OSp(4|4)$ in $PSU(4|4)$.

2.1 Basic Examples

It is convenient to split the scalars Φ_i, $i = 1,\ldots,6$ into two groups acted on respectively by the two factors of $SO(3) \times SO(3) \subset SO(6)_R$. We take these two groups to consist of the first three and last three Φ's; we rename (Φ_1,Φ_2,Φ_3) as $\vec{X} = (X_1,X_2,X_3)$ and (Φ_4,Φ_5,Φ_6) as $\vec{Y} = (Y_1,Y_2,Y_3)$. We sometimes write $SO(3)_X$ and $SO(3)_Y$ for the two $SO(3)$ groups.

Though the **16** of $SO(1,9)$, in which the supersymmetries transform, is irreducible, it is as already explained reducible as a representation of $W = SO(1,2) \times SO(3)_X \times SO(3)_Y$. Indeed, the action of W commutes with the three operators

$$\begin{aligned} B_0 &= \Gamma_{456789}, \\ B_1 &= \Gamma_{3456}, \\ B_2 &= \Gamma_{3789}. \end{aligned} \tag{2.7}$$

They obey $B_0^2 = -1$, $B_1^2 = B_2^2 = 1$, and $B_0B_1 = -B_1B_0 = B_2$, etc., and generate an action of $SL(2,\mathbb{R})$. We can decompose the **16** of $SO(1,9)$ as $V_8 \otimes V_2$, where V_8 transforms in the real irreducible representation $(\mathbf{2},\mathbf{2},\mathbf{2})$ of $SO(1,2) \times SO(3)_X \times SO(3)_Y$, and V_2 is a two-dimensional space in which the B_i are represented by

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (2.8)$$
$$B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A boundary condition preserves supersymmetry if and only if it ensures that the component of the supercurrent normal to the boundary vanishes. The supercurrent was written in (2.2). For a supersymmetry generator ε, the condition we need is that

$$\mathrm{Tr}\,\overline{\varepsilon}\Gamma^{IJ}F_{IJ}\Gamma_3\Psi = 0. \qquad (2.9)$$

For a half-BPS boundary condition, we do not expect this to hold for all ε, but only for ε in a middle-dimensional subspace of $V_8 \otimes V_2$. In fact, to achieve $OSp(4|4)$ invariance, the condition must hold precisely for $\varepsilon = v \otimes \varepsilon_0$, where ε_0 is a fixed element of V_2 and v is an arbitrary element of V_8. The choice of ε_0 is equivalent to a choice of $OSp(4|4)$ embedding in $PSU(4|4)$.

The expression $(\varepsilon, \tilde{\varepsilon}) = \overline{\varepsilon}\Gamma_3\tilde{\varepsilon}$ defines an $SO(1,2)\times SO(6)$-invariant quadratic form on the **16** of $SO(1,9)$. For $\varepsilon = v \otimes \varepsilon_0$, $\tilde{\varepsilon} = \tilde{v} \otimes \tilde{\varepsilon}_0$, we have $(\varepsilon, \tilde{\varepsilon}) = \langle v, \tilde{v}\rangle\langle \varepsilon_0, \tilde{\varepsilon}_0\rangle$, where the two factors are antisymmetric inner products on V_8 and on V_2. If we think of ε_0 as a column vector $\binom{s}{t}$ and $\overline{\varepsilon}_0$ as the row vector $(t \; -s)$, then we can write the inner product on V_2 as $\langle \varepsilon_0, \tilde{\varepsilon}\rangle = \overline{\varepsilon}_0\tilde{\varepsilon}_0$.

What boundary conditions should we impose on Ψ and the bosonic fields? In general, a local boundary condition for fermions sets to zero half the components of the fermions. For invariance under $W = SO(1,2)\times SO(3)\times SO(3)$, the boundary condition on Ψ must be that $\Gamma_3\Psi = \Psi' \otimes \vartheta$, where Ψ' takes values in[2] $V_8 \otimes \mathfrak{g}$ ($\mathfrak{g}$ is the Lie algebra of G) and ϑ is a fixed vector in V_2. Note that, as Γ_3 reverses the ten-dimensional chirality, we have

$$\overline{\Gamma}\Psi' = -\Psi'. \qquad (2.10)$$

Equation (2.9) is equivalent to

$$\begin{aligned}
0 &= \overline{\varepsilon}(\Gamma^{\mu\nu}F_{\mu\nu} + 2\Gamma^{3\mu}F_{3\mu})\Psi'\vartheta,\\
0 &= \sum_{\mu=0,1,2} \overline{\varepsilon}(\Gamma^{\mu a}D_\mu X_a)\Psi'\vartheta,\\
0 &= \sum_{\mu=0,1,2} \overline{\varepsilon}(\Gamma^{\mu m}D_\mu Y_m)\Psi'\vartheta,\\
0 &= \overline{\varepsilon}\Gamma^{am}[X_a, Y_m]\Psi'\vartheta,\\
0 &= \overline{\varepsilon}(2\Gamma^{3a}D_3X_a + \Gamma^{ab}[X_a, X_b])\Psi'\vartheta,\\
0 &= \overline{\varepsilon}(2\Gamma^{3m}D_3Y_a + \Gamma^{mn}[Y_m, Y_n])\Psi'\vartheta.
\end{aligned} \qquad (2.11)$$

Here Greek indices μ, ν originate from ten-dimensional indices $0, 1, 2$, while indices a, b, c labeling X and indices m, n, p labeling Y originate from ten-dimensional indices $4, 5, 6$

[2] Ψ takes values in the **16** of $SO(1,9)$ and $\Gamma_3\Psi$ in the **16**$'$. Multiplication by Γ_{012} exchanges these spaces while commuting with $SO(1,2)\times SO(3)_X \times SO(3)_Y$ and with the B's. So for our purposes, we can identify them both as $V_8 \otimes V_2$.

and 7, 8, 9, respectively. Summations over all relevant values are understood except where indicated. We must pick ε_0 and ϑ as well as the boundary conditions obeyed by the bosonic fields to ensure these equations.

Writing $\overline{\varepsilon} = \overline{v} \otimes \overline{\varepsilon}_0$, we want to eliminate $\overline{v}$ and Ψ' and write these equations just in terms of $\overline{\varepsilon}_0$ and ϑ. To do this in the first equation, we write $\Gamma^{3\mu} = -\frac{1}{2}\epsilon^{\mu\nu\lambda}\Gamma_{\nu\lambda}\Gamma_{0123}$, where $\epsilon^{\mu\nu\lambda}$ is the antisymmetric tensor in $\mathbb{R}^{1,2}$ (with $\epsilon^{012} = 1$). Then, using (2.10), we can replace $\Gamma_{0123}\Psi'$ by $B_0\Psi'$. At this point, the first equation in (2.11) reduces to $\overline{\varepsilon}_0(F_{\mu\nu} - \epsilon_{\mu\nu\lambda}F^{3\lambda}B_0)\vartheta = 0$. To similarly rewrite the second equation, we want to replace $\Gamma^{\mu a}$ with the product of a matrix that acts in V_8 and one that acts in V_2. We do this via $(\Gamma^{\mu a})\Psi' = -\frac{1}{4}(\epsilon^{\mu\nu\lambda}\epsilon^{abc}\Gamma_{\nu\lambda}\Gamma_{bc}B_2)\Psi'$, where (2.10) has been used. With similar manipulations, we can write each equation just in terms of $\overline{\varepsilon}_0$ and ϑ:

$$\begin{aligned}
0 &= \overline{\varepsilon}_0(F_{\mu\nu} - \epsilon_{\mu\nu\lambda}F^{3\lambda}B_0)\cdot\vartheta,\\
0 &= D_\mu X_a \cdot \overline{\varepsilon}_0 B_2\vartheta,\\
0 &= D_\mu Y_m \cdot \overline{\varepsilon}_0 B_1\vartheta,\\
0 &= [X_a, Y_m]\cdot \overline{\varepsilon}_0 B_0\vartheta,\\
0 &= \overline{\varepsilon}_0([X_b, X_c] - \epsilon_{abc}D_3X_aB_1)\vartheta,\\
0 &= \overline{\varepsilon}_0([Y_m, Y_n] - \epsilon_{pmn}D_3Y_pB_2)\vartheta.
\end{aligned} \tag{2.12}$$

(All expressions are to be evaluated at $y = 0$.) In analyzing these equations, we will at first consider only boundary conditions that preserve the full gauge symmetry.

To satisfy the first equation, we have to assume that the boundary condition for the gauge fields is

$$\epsilon_{\lambda\mu\nu}F^{3\lambda} + \gamma F_{\mu\nu} = 0, \tag{2.13}$$

where γ is a constant (γ equals 0 for the usual Neumann boundary condition $F_{3\lambda} = 0$ and ∞ for Dirichlet boundary conditions $F_{\mu\nu} = 0$, $\mu, \nu \neq 3$). Then in addition, we must choose ε_0 and ϑ so that

$$\overline{\varepsilon}_0(1 + \gamma B_0)\vartheta = 0. \tag{2.14}$$

The alternative of satisfying the first equation in (2.12) by setting $\overline{\varepsilon}_0\vartheta = \overline{\varepsilon}_0 B_0\vartheta = 0$ is not viable, since it cannot be satisfied for real ε_0.

The nature of the remaining equations depends on whether X or Y or both obeys Dirichlet boundary conditions or in other words is required to vanish on the boundary. If we place Dirichlet boundary conditions on neither X nor Y, then to obey the second, third, and fourth equations we need $0 = \overline{\varepsilon}_0 B_0\vartheta = \overline{\varepsilon}_0 B_1\vartheta = \overline{\varepsilon}_0 B_2\vartheta$. But these conditions are overdetermined and force $\vartheta = \varepsilon_0 = 0$.

If we place Dirichlet boundary conditions on both X and Y, then the second, third, and fourth equations become trivial. However, the last two equations give $\overline{\varepsilon}_0 B_1\vartheta = \overline{\varepsilon}_0 B_2\vartheta = 0$. These equations have no nonzero solution with real ε_0, so also this case does not occur.

What remains is the case of Dirichlet boundary conditions on just one of X and Y. Of course, the two cases are equivalent. For definiteness, we assume that Y obeys Dirichlet boundary conditions. If we take the boundary condition on X to be

$$D_3X_a + \frac{u}{2}\epsilon_{abc}[X_b, X_c] = 0 \tag{2.15}$$

for some constant u, then all equations are satisfied if

$$0=\overline{\varepsilon}_0 B_2\vartheta=\overline{\varepsilon}_0\,(1+uB_1)\,\vartheta. \tag{2.16}$$

Equations (2.14) and (2.16) enable us to determine everything in terms of $\overline{\varepsilon}_0$, the assumed generator of the unbroken supersymmetry. Let us write $\overline{\varepsilon}_0$ as a row vector; by scaling we can put it in the form $\overline{\varepsilon}_0=(1a)$. Then viewing ϑ as a column vector, we find that up to scaling

$$\vartheta=\begin{pmatrix}a\\1\end{pmatrix}. \tag{2.17}$$

Moreover,

$$\gamma=-\frac{2a}{1-a^2},\qquad u=-\frac{2a}{1+a^2}. \tag{2.18}$$

Both γ and u change sign under $a\to -a$. This results from the action on the boundary conditions of a reflection symmetry of the underlying super Yang-Mills theory. The symmetry acts by a reflection of one of the spatial coordinates parallel to the boundary, say x^1, and a sign change of X. A reflection of x^1 with a sign change of Y rather than X corresponds to $a\to 1/a$, $\gamma\to-\gamma$, $u\to u$, which is also a symmetry of the above formulas.

2.1.1 Interpretation

Let us now discuss the interpretation of some of these boundary conditions.

NS5-Like Boundary Condition The first important case arises if ε is an eigenvector of B_2, or equivalently if $a=0$ or ∞. Then γ and u vanish, meaning that the scalar fields X and the three-dimensional gauge field A_μ, $\mu=0,1,2$ obey Neumann boundary conditions. They combine together from a three-dimensional point of view into a vector multiplet. (This statement is explained more fully in Sect. 2.3.) Meanwhile, Y and A_3 combine to a hypermultiplet in the three-dimensional sense; it is subject to Dirichlet boundary conditions. In fact, for $G=U(N)$, these are the boundary conditions that arise for parallel D3-branes ending on a single NS5-brane whose world-volume is parametrized by x^0, x^1, x^2 and x^4, x^5, x^6 (with the four-dimensional θ-angle vanishing). We refer to boundary conditions that preserve such supersymmetry as NS5-like.

D5-Like Boundary Condition A second important case is that ε is an eigenvector of B_1, or $a=\pm 1$. Then γ is infinite, which means that the gauge field obeys Dirichlet boundary conditions, with $F_{\mu\nu}$ vanishing on the boundary for $\mu,\nu=0,1,2$. Y also obeys Dirichlet boundary conditions. Indeed, at $a=\pm1$, A_μ and Y are a vector multiplet from a three-dimensional point of view. The hypermultiplet is described by X and A_3, and obeys modified Neumann boundary conditions, with $u=\pm1$ in (2.15). These rather simple boundary conditions preserve the same supersymmetry of a system of D3-branes ending on a D5-brane (with the same world-volume as the NS5-brane in the last paragraph), and we call them D5-like. But as we discuss in Sect. 3.4, they do not correspond to the case of D3-branes ending on a *single* D5-brane.

One simple but important point is that the Dirichlet boundary conditions for $\vec{Y}$ can be slightly generalized (in some cases, this generalization can be realized in string theory by displacing branes in the $\vec{Y}$ direction). Instead of taking $\vec{Y}$ simply to vanish, we can pick any commuting triple $\vec{w} \in \mathfrak{g}$ (that is, any three elements $w_m \in \mathfrak{g}$ such that $[w_m, w_n] = 0$) and take the boundary condition to be

$$\vec{Y}(0) = \vec{w}. \tag{2.19}$$

Because we take $\vec{w}$ to be *constant* (independent of the spatial coordinates) and because the gauge field A_μ vanishes on the boundary, this gives no contribution to the $D_\mu Y_m$ term in the boundary constraint (2.12). Because the components of $\vec{w}$ commute, there is no contribution to the $[Y_m, Y_n]$ term, and because $\overline{\varepsilon} B_0 \vartheta = 0$ for D5-like supersymmetry, the $[X_a, Y_m]$ term is harmless. This establishes the supersymmetry of (2.19).

The θ Angle Finally, let us consider the case of generic a. The general conformally-invariant boundary condition (2.13) for the gauge fields, which says that on the boundary the normal part of the field strength is a prescribed multiple of the tangential part, is the natural extension of Neumann boundary conditions for gauge fields in the presence of a four-dimensional θ-angle. If one adds the θ-term to the usual Yang-Mills action, so that the combined action takes the form

$$I = \frac{1}{e^2} \int d^4x \, \mathrm{Tr}\left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu}\right) + \frac{\theta}{8\pi^2} \int \mathrm{Tr}\, F \wedge F, \tag{2.20}$$

then upon varying I with respect to A, with no restriction on the variation of A at the boundary, one arrives at the boundary condition of (2.13) with $\gamma = -\theta e^2/4\pi^2$.

2.2 Boundary Conditions that Reduce the Gauge Symmetry

The boundary conditions constructed in Sect. 2.1 preserved the full gauge symmetry. It turns out, however, that there are also half-BPS boundary conditions that break part of the gauge symmetry. Since this idea may seem strange at first, we motivate it by starting with a natural special case, which arises in string theory for D3-branes ending on an orientifold or orbifold five-plane. We will present this construction for D5-type supersymmetry (which arises for an orientifold plane in the 012456 directions or an orbifold that involves reflection of directions 3789). Or course, by exchanging $\vec{X}$ and $\vec{Y}$, one can make a similar construction for NS5-like supersymmetry.

Instead of formulating the discussion in terms of a boundary condition, we start with $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^{1,3}$, with no restriction on the sign of x^3. However, we require that all fields are invariant under a reflection $x^3 \to -x^3$, combined with a suitable automorphism. Field theory on $\mathbb{R}^{1,3}$ with this symmetry imposed is equivalent to field theory on the half-space $x^3 \geq 0$ with a suitable boundary condition. The advantage of working on the covering space is that it makes it more obvious how to reduce the gauge symmetry while preserving supersymmetry.

To get a symmetry of $\mathcal{N} = 4$ super Yang-Mills theory, a reflection of space must be accompanied by a reflection of an odd number of the scalar fields Φ^i (so as to preserve the orientation of the underlying ten-dimensional spacetime $\mathbb{R}^{1,9}$). To preserve supersymmetry, it is necessary to reflect precisely three[3] of the Φ^i. To in addition preserve the standard $SO(3) \times SO(3)$ R-symmetry (rather than a group conjugate to this), we choose to reflect $\vec{X}$ and not $\vec{Y}$, or vice-versa.

[3] The total number of reflected coordinates, including x^3, is then 4. This is compatible with supersymmetry since for instance $(\Gamma_{3789})^2 = 1$.

In any event, we also accompany these reflections with an automorphism τ of the gauge group G. τ must obey $\tau^2=1$ and may be either an inner automorphism or an outer automorphism. Both cases can be realized in string theory with D3-branes, by using certain orbifolds or orientifolds for inner or outer automorphisms. This will be discussed in detail elsewhere. Here, we simply work in field theory.

It is convenient to decompose the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g}=\mathfrak{g}^+\oplus\mathfrak{g}^-$, where τ acts on $\mathfrak{g}^\pm$ by multiplication by ± 1. For any adjoint-valued field Φ, we write $\Phi=\Phi^++\Phi^-$, where $\Phi^\pm$ take values in $\mathfrak{g}^\pm$. We also write Φ^τ for $\tau\Phi\tau^{-1}$. We require that all fields should be invariant under the action of τ combined with a reflection of x^3,x^7,x^8,x^9:

$$\begin{aligned} A_\mu(x^3)&=A_\mu^\tau(-x^3),\quad \mu=0,1,2\\ A_3(x^3)&=-A_3^\tau(-x^3),\\ \vec{X}(x^3)&=-\vec{X}^\tau(-x^3),\\ \vec{Y}(x^3)&=\vec{Y}^\tau(-x^3). \end{aligned}\tag{2.21}$$

This implies certain conditions on the behavior at the fixed plane $x^3=0$. Writing $\Phi|$ for the restriction of a field Φ to $x^3=0$, we get

$$\begin{aligned} F_{3\mu}^+|&=F_{\mu\nu}^-|=0,\\ D_3X^-|&=X^+|=0,\\ Y^-|&=D_3Y^+|=0. \end{aligned}\tag{2.22}$$

To describe the boundary conditions on the fermions, we write $\Psi'=\psi^+\otimes\vartheta^++\psi^-\otimes\vartheta^-$, where $\psi^\pm$ is valued in $V_8\otimes\mathfrak{g}^\pm$, and $\vartheta^\pm$ is valued in V_2. By imitating the steps that led to (2.12), (2.14), and (2.16), one now finds that the condition for maintaining one half of the supersymmetry is that

$$\begin{aligned} \overline{\varepsilon}_0\vartheta^+&=\overline{\varepsilon}_0B_1\vartheta^+=0,\\ \overline{\varepsilon}_0B_0\vartheta^-&=\overline{\varepsilon}_0B_2\vartheta^-=0. \end{aligned}\tag{2.23}$$

These conditions are equivalent to $\overline{\varepsilon}_0B_1=w\overline{\varepsilon}_0$, $B_1\vartheta^\pm=\mp w\vartheta^\pm$, where $w=\pm 1$; the two choices of w are equivalent under a reflection (say $x^1\to -x^1$) that acts trivially on x^3 and reverses the sign of $\vec{X}$. Since the eigenspaces of B_1 are one-dimensional, everything is determined up to scaling once w is chosen.

The two choices of w correspond to $a=0,\infty$; equivalently, ε_0 is an eigenvector of B_1. The above boundary condition is D5-like in the sense of Sect. 2.1.1. In fact, if $G=U(1)$ and τ is the complex conjugation operation that acts on the Lie algebra as multiplication by -1 (thus, τ is "charge conjugation"), then the above is the standard Dirichlet or D5-like boundary condition—Dirichlet for A_μ and $\vec{Y}$, Neumann for $\vec{X}$ and A_3. Since multiplication by -1 is not a symmetry of a nonabelian Lie algebra, one might be puzzled what is the analog of this statement for nonabelian G. That will become clear in Sect. 2.2.1.

Alternatively, if we set $\tau=1$ and exchange $\vec{X}$ and $\vec{Y}$, we get the simplest NS5-like boundary condition of Sect. 2.1.1.

2.2.1 Generalization to Any H

The above construction has a generalization that may appear surprising at first sight (but whose existence may become more obvious in Sect. 2.3.3).

In the derivation, we decomposed $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, where $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are even and odd under τ. Of course, $\mathfrak{g}^+$ is a Lie algebra—it is the Lie algebra of the subgroup H of G that commutes with τ. Normally, $\mathfrak{g}^-$ is not a Lie algebra. In general, we have

$$[\mathfrak{g}^+, \mathfrak{g}^+] = \mathfrak{g}^+, \qquad [\mathfrak{g}^+, \mathfrak{g}^-] = \mathfrak{g}^-, \qquad [\mathfrak{g}^-, \mathfrak{g}^-] = \mathfrak{g}^+, \tag{2.24}$$

expressing the fact that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are respectively even and odd under τ. The first equation asserts that $\mathfrak{g}^+$ is a Lie algebra. The second asserts that $\mathfrak{g}^-$ furnishes a representation of this Lie algebra. The third equation asserts that H is a very special type of subgroup of G: the quotient G/H is a symmetric space.

A close examination of the verification of the supersymmetry of the boundary conditions of (2.22) shows that while the first two conditions in (2.24) are needed, the third is not. Therefore, we can generalize the above construction to the case of a general subgroup $H \subset G$, not necessarily related to a homogeneous space. What we will get this way can no longer be interpreted as the result of imposing reflection symmetry on gauge theory on $\mathbb{R}^{1,3}$. But it will still give a half-BPS boundary condition for gauge theory on the half-space.

In detail, we proceed as follows. We pick an *arbitrary* subgroup[4] H of G, and decompose the Lie algebra of G as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h}$ is the Lie algebra of H, and $\mathfrak{h}^\perp$ is its orthocomplement. For any adjoint-valued field Φ, we write $\Phi = \Phi^+ + \Phi^-$, where $\Phi^+ \in \mathfrak{h}$, $\Phi^- \in \mathfrak{h}^\perp$. Now we formulate $\mathcal{N} = 4$ super Yang-Mills theory on the half-space $x^3 \geq 0$, restricting some fields (or their normal derivatives) to $\mathfrak{h}$ and some to $\mathfrak{h}^\perp$, according to (2.22). In this way, we get a half-BPS boundary condition in which the gauge group is reduced along the boundary from G to H, for any $H \subset G$. (In quantizing the theory, we divide by gauge transformations that are H-valued along the boundary.) Of course, by exchanging $\vec{X}$ and $\vec{Y}$, we get a second such boundary condition. Of these two boundary conditions, the first is D5-like and the second is NS5-like.

An important special case is the case that H is the trivial subgroup of G, consisting only of the identity element. Then $\mathfrak{g}^+ = 0$ and $\mathfrak{g}^- = \mathfrak{g}$; so for any field Φ, we have $\Phi^+ = 0$, $\Phi^- = \Phi$. Then (2.22) reduces to standard Dirichlet boundary conditions (that is, Dirichlet for A_μ and $\vec{Y}$, Neumann for $\vec{X}$ and A_3).

2.2.2 Global Symmetries

An important property of boundary conditions with reduced gauge symmetry is that they may admit global symmetries. Let K be the subgroup of G that commutes with H. The boundary conditions just described, in which G is reduced to H along the boundary, admit constant gauge transformations by an element of K. These behave as global symmetries, since at the boundary they are not equivalent to gauge transformations. A local operator at $y \neq 0$ is required to be G-invariant, and so in particular K-invariant, but a local operator at $y = 0$ is only required to be H-invariant. So in particular, local operators that transform non-trivially under K exist at and only at $y = 0$. The S-dual of this situation involves a construction that we will explain in Sect. 2.3: for NS-like boundary conditions, it is possible to introduce matter fields supported only at the boundary. These may carry global symmetries, and naturally local operators that transform non-trivially under those symmetries exist only on the boundary.

A special case is that if $H = 1$ is the trivial group with only the identity element, then K is all of G. In this case, G acts by global symmetries on the boundary.

[4] In most of this paper, our considerations are local and only the connected component of H is relevant.

The boundary condition with $H=1$ is actually not exotic at all. It coincides with the basic D5-like boundary conditions in which the vector multiplet obeys Dirichlet boundary conditions and the hypermultiplet obeys Neumann boundary conditions. If $H=1$, then for any field Φ, we have $\Phi^+=0$ and $\Phi^-=\Phi$. As a result, the boundary conditions (2.22) are equivalent to the D5-like boundary conditions summarized in Sect. 2.1.1.

2.2.3 Central Elements

In (2.22), we have placed Dirichlet boundary conditions on both $\vec{X}^+$ and $\vec{Y}^-$. Just as in our earlier treatment of (2.19), these conditions can be slightly generalized[5] so that the boundary values of the fields in question are constant, but not zero. (This generalization will typically break some of the global symmetries that were just described.)

First of all, we let $\mathcal{Z}(\mathfrak{g}^+)$ denote the center of $\mathfrak{g}^+$, and we let $\mathcal{Z}(\mathfrak{g}^-)$ denote the subspace of $\mathfrak{g}^-$ that commutes with $\mathfrak{g}^+$. Let $\vec{v}$ and $\vec{w}$ be triples of elements of $\mathcal{Z}(\mathfrak{g}^+)$ and $\mathcal{Z}(\mathfrak{g}^-)$, respectively, such that the components of $\vec{w}$ commute with each other. The components of $\vec{v}$ automatically commute with each other since $\mathcal{Z}(\mathfrak{g}^+)$ is abelian, and the components of $\vec{w}$ commute with those of $\vec{v}$ since $\vec{w}$ commutes with $\mathfrak{g}^+$, which contains $\vec{v}$. So in fact all components of $\vec{v}$ and $\vec{w}$ commute.

Then without breaking supersymmetry, the simple Dirichlet boundary conditions $\vec{X}^+(0)=\vec{Y}^-(0)=0$ can be replaced by

$$\begin{aligned}\vec{X}^+(0)&=\vec{v},\\ \vec{Y}^-(0)&=\vec{w}.\end{aligned}\tag{2.25}$$

Indeed, using (2.23) and the fact that all components of $\vec{v}$ and $\vec{w}$ commute with each other and with $A_\mu(0)$, one can verify the vanishing of all contributions to (2.12) that depend on $\vec{v}$ or $\vec{w}$.

2.3 Coupling the NS System to Matter

We have constructed quite a few half-BPS boundary conditions, but nonetheless an attempt to understand the action of electric-magnetic duality on the boundary conditions we have seen so far would fail. Generically, duality maps boundary conditions that we have described to ones that we have not yet described. We explain an important extension for the NS5 case here and an important extension for the D5 case in Sect. 2.4.1. It will turn out that these two extensions make it possible to describe the action of S-duality (though in this paper we take only preliminary steps in that direction).

We begin with the NS5-like boundary condition summarized in Sect. 2.1.1, in which A_μ and three scalars obey Neumann boundary conditions, while A_3 and the other three scalars obey Dirichlet boundary conditions. However, we will make a small change of notation from Sect. 2.1. In that section, we considered a one-parameter family of possible choices of the unbroken supersymmetry, always denoting as $\vec{Y}$ the scalars that obey Dirichlet boundary conditions. The parameter that enters the choice of supersymmetry is important, and we further explore its role elsewhere [10]. But in the rest of the present paper, we will consider only boundary conditions that have the same supersymmetry as the D3-D5 system, or equivalently, if we exchange $\vec{X}$ and $\vec{Y}$, the same supersymmetry as the D3-NS5 system.

[5] Equation (2.19) is equivalent to the special case of what follows in which H is trivial, $\mathfrak{g}^+=0$ and $\mathfrak{g}^-=\mathfrak{g}$.

We will describe several different constructions, and will want to combine them together. This is more straightforward if they all preserve the same supersymmetry. So in the rest of this paper, we always assume that the generator $\overline{\varepsilon}$ of the unbroken supersymmetry is an eigenvector of B_1. A related statement is that, in a sense that will become clear, though we will consider many different boundary conditions for vector multiplets and hypermultiplets, in the rest of this paper, $\vec{X}$ will always transform in a hypermultiplet and $\vec{Y}$ will always be part of a vector multiplet.

To put in this framework the simplest NS5-like boundary conditions, we make a change of notation relative to Sect. 2.1, and exchange $\vec{X}$ and $\vec{Y}$. Thus, the boundary conditions that we will generalize, without changing the unbroken supersymmetry, are Neumann boundary conditions for A_μ and $\vec{Y}$, together with Dirichlet boundary conditions for $\vec{X}$, suitably extended to the rest of the supermultiplet.

2.3.1 Three-Dimensional Theory with Infinite-Dimensional Gauge Group

In particular, in our starting point, at the boundary $y = 0$ there are gauge fields of the full G symmetry. This being so, one can introduce additional degrees of freedom that carry the G symmetry and are supported at the boundary. These additional degrees of freedom must have $\mathcal{N} = 4$ superconformal symmetry if the combined system is to have that property, but otherwise they are arbitrary.

Of course, we should ask here whether a bulk system with $\mathcal{N} = 4$ supersymmetry (in the four-dimensional sense) can be coupled to a boundary system with $\mathcal{N} = 4$ supersymmetry (in the three-dimensional sense), in such a way as to preserve the full supersymmetry of the boundary theory. A rather similar question, involving defects instead of boundaries, was addressed in reference [6]. Rather than performing a similar calculation, we will take a short-cut, first of all to show that the supersymmetric coupling exists at the classical level. We will assume to begin with that the boundary theory is described by hypermultiplets that parametrize a hyper-Kahler manifold Z with G symmetry.

The first step will be to describe the gauge theory on the half-space $y \geq 0$ as a three-dimensional theory with an infinite-dimensional gauge group. We let L be the half-line $y \geq 0$, and we think of the half-space $y \geq 0$ as $\mathbb{R}^{1,2} \times L$. We let $\hat{G}$ be the group of maps from L to G. The Lie algebra of $\hat{G}$ is spanned by $\mathfrak{g}$-valued functions on L. On this Lie algebra, there is a natural positive definite inner product; if a and b are two such functions, we define $\langle a, b\rangle = -\int \mathrm{d}y \,\mathrm{Tr}\, ab$, where $-\mathrm{Tr}\, ab$ is a positive definite invariant inner product on L. So formally we can write down in the usual way a supersymmetric gauge theory action on $\mathbb{R}^{1,2}$, with $\mathcal{N} = 4$ supersymmetry in the three-dimensional sense, for a vector multiplet with gauge group $\hat{G}$. The fields in this theory are the three-dimensional gauge field A_μ, $\mu = 0, 1, 2$ (but not A_3), plus the scalars $\vec{Y}$ (but not $\vec{X}$), and half of the fermions of $\mathcal{N} = 4$ super Yang-Mills theory.

This theory, though formally supersymmetric, is not really well-behaved unless we also add suitable hypermultiplets. The reason is that the kinetic energy contains no derivatives in the y direction. For example, the gauge theory part of the action is

$$\frac{1}{2e^2}\int_{\mathbb{R}^{1,2}} \mathrm{d}^3x \int_L \mathrm{d}y \sum_{\mu,\nu=0,1,2} \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}. \tag{2.26}$$

Here the integral over $\mathbb{R}^{1,2}$ is part of the definition of three-dimensional gauge theory, and the integral over L arises because it is part of the definition of the quadratic form on the Lie algebra. Clearly, (2.26) is part of the usual Yang-Mills action in four dimensions, but the terms involving $F_{3\mu}$ and containing derivatives in the y direction are missing.

To complete the theory, we need hypermultiplets, namely the additional fields A_3 and $\vec{X}$. They parametrize an infinite-dimensional flat hyper-Kahler manifold. The hyper-Kahler metric is

$$ds^2 = -\int_L \mathrm{d}y \,\mathrm{Tr}\left(\delta A_3^2 + \sum_i \delta X_i^2\right). \tag{2.27}$$

The three hyper-Kahler forms are

$$\omega_i = \int_L \mathrm{d}y \,\mathrm{Tr}(\delta A_3 \wedge \delta X_i + \delta X_{i+1} \wedge \delta X_{i-1}), \quad i = 1, 2, 3, \tag{2.28}$$

where we set $X_{i+3} = X_i$. This formula is covariant under $SO(3)$ rotations of X_i and ω_i, though not written so as to make this manifest.

These equations describe an infinite-dimensional flat hyper-Kahler manifold on which $\hat{G}$ acts by gauge transformations. One point to mention here is that the fields X_a transform in the adjoint representation of $\hat{G}$, but A_3, because of its inhomogeneous gauge transformation law $\delta A_3 = -D_3 u = [u, A_3] - \partial_3 u$ (where u is the generator of a gauge transformation), transforms in what one might call an "affine deformation" of the adjoint representation. This has no close analog for finite-dimensional groups.

Nonetheless, the pair $(\vec{X}, A_3)$ form a hypermultiplet, that is, they parametrize a hyper-Kahler manifold with $\hat{G}$ action. So following the standard recipe, we can formally write down the three-dimensional supersymmetric action for the coupling of this hyper-Kahler manifold to the vector multiplet of $\hat{G}$. The sum of this action with the vector multiplet action described earlier is the action of four-dimensional $\mathcal{N}=4$ super Yang-Mills theory on the half-space. For example, the kinetic energy of the hypermultiplet gives the $F_{3\mu}^2$ term that was missing in (2.26).

From this point of view, there is no problem to add additional hypermultiplets, with G symmetry, that are supported at $y=0$. First of all, there is a natural homomorphism from $\hat{G}$ to G by evaluation at $y=0$. Thus, if $g(y) : L \to G$ is an element of $\hat{G}$, we simply map $g(y)$ to its boundary value $g(0)$. So if Z is any space with G symmetry, we can regard it as a space with $\hat{G}$ symmetry: an element $g(y) \in \hat{G}$ acts on Z via the given action of $g(0)$. If therefore Z is a hyper-Kahler manifold with G action, we can view it as a hyper-Kahler manifold with $\hat{G}$ action. Then we just write down the standard $\mathcal{N}=4$ theory in the three-dimensional sense, with the vector multiplets being those of the group $\hat{G}$, and the hypermultiplets being $(A_3, \vec{X})$ and the fields parametrizing Z.

This construction gives a four-dimensional theory with a boundary hypermultiplet. The theory is conformally invariant at the classical level if and only if the purely three-dimensional theory with target Z is conformally invariant. In turn, that is so precisely if the hyper-Kahler manifold Z is conical, for example if Z is a linear manifold $\mathbb{R}^{4n}$ for some n.

It is also possible to modify this construction by taking the metric on the Lie algebra of $\hat{G}$ to be $\langle a, b\rangle = -\int \mathrm{d}y e(y)^{-2}\,\mathrm{Tr}\, ab$, with an arbitrary positive definite function $e(y)^2$. This gives a construction of the half-BPS Janus configuration, first described in field theory in [11], for the case that the gauge coupling e is a function of y but the angle θ is constant. For the generalization to varying θ, see [10].

2.3.2 Shifted Boundary Condition of $\vec{Y}$

By computing the hyper-Kahler moment map of $(A_3, \vec{X})$, we can get a new understanding of some known results about coupling of bulk gauge fields to localized hypermultiplets

[6]. To compute the hyper-Kahler moment map, we must contract ω_i with the vector fields $\delta A_3 = -D_3\alpha$, $\delta X_i = [\alpha, X_i]$ that generate the action of the gauge group. We call this vector field $V(\alpha)$. Its contraction with ω_i is

$$\iota_{V(\alpha)}\omega_i = \int \mathrm{d}y\,\mathrm{Tr}(-D_3\alpha\delta X_i - \delta A_3[\alpha, X_i] + \alpha[X_{i+1}, \delta X_{i-1}] - \alpha[\delta X_{i+1}, X_{i-1}]). \tag{2.29}$$

The definition of the hyper-Kahler moment map $\mu_i(\alpha)$ is that $\delta\mu_i(\alpha) = \iota_{V(\alpha)}\omega_i$. A short calculation, with some integration by parts, shows that

$$\mu_i(\alpha) = \int \mathrm{d}y\,\mathrm{Tr}\left(\alpha\left(\frac{DX_i}{Dy} + [X_{i+1}, X_{i-1}]\right)\right) + \mathrm{Tr}\,\alpha X_i(0). \tag{2.30}$$

In integrating by parts, we have included a surface term at $y = 0$, but a possible surface term at $y = \infty$ vanishes if the energy is finite and will not be important.

The consequences of this formula may be clearer if instead of writing the pairing of the moment map $\vec{\mu}$ with an arbitrary element α of the Lie algebra of $\hat{G}$, we write out $\vec{\mu}$ as a $\mathfrak{g}$-valued function on L:

$$\vec{\mu}(y) = \frac{D\vec{X}}{Dy} + \vec{X} \times \vec{X}(y) + \delta(y)\vec{X}(0). \tag{2.31}$$

Now we can get a somewhat better understanding of the NS boundary condition summarized in Sect. 2.1.1. In general, for coupling to any hypermultiplets, the action contains a term $\int \mathrm{d}^3x(\vec{\mu}, \vec{\mu})$. In the present context, this means $-\int_{\mathbb{R}^{2,1}} \mathrm{d}^3x \int_L \mathrm{d}y\,\mathrm{Tr}\,\vec{\mu}^2$. Because of the delta function in $\vec{\mu}$, the action is finite only if $\vec{X}(0) = 0$, which (modulo the exchange of $\vec{Y}$ and $\vec{X}$) is the boundary condition that we found in Sect. 2.1.

Now we can generalize this to the case that a boundary hypermultiplet is present, parametrizing a hyper-Kahler manifold Z. Z has its own hyper-Kahler moment map $\vec{\mu}^Z$, and the hyper-Kahler moment map of the combined system is obtained by adding this to (2.31):

$$\vec{\mu}(y) = \frac{D\vec{X}}{Dy} + \vec{X} \times \vec{X}(y) + \delta(y)(\vec{X}(0) + \vec{\mu}^Z). \tag{2.32}$$

To keep the action finite, it now must be that in the presence of the boundary hypermultiplet, the boundary condition on $\vec{X}$ is shifted from $\vec{X}(0) = 0$ to

$$\vec{X}(0) + \vec{\mu}^Z = 0. \tag{2.33}$$

This closely parallels a result in [6].

2.3.3 Analog for General H

We can now get a new understanding of the boundary conditions found in Sect. 2.2.1 with the gauge symmetry reduced from G to H along the boundary.

For any subgroup H of G, we define a subgroup $\hat{G}_H$ of $\hat{G}$ that consists of maps $g : L \to G$ such that $g(0) \in H$. We take $(A_\mu, \vec{Y})$ to be the vector multiplets of a three-dimensional theory with gauge group $\hat{G}_H$. And we interpret $(A_3, \vec{X})$ as hypermultiplets of this symmetry, valued in the adjoint representation but with the boundary condition that

$\vec{X}(0)$ is valued in $\mathfrak{h}^{\perp}$. As above, the condition on $\vec{X}(0)$ can be explained by computing the delta function contribution to the moment map, which turns out to be the projection of $\vec{X}(0)$ to $\mathfrak{h}$ (the projection arises simply because the Lie algebra of $\hat{G}_H$ is spanned by functions $\alpha : L \to \mathfrak{g}$ with $\alpha(0) \in \mathfrak{h}$).

The $\mathcal{N}=4$ supersymmetric theory with this vector multiplet and hypermultiplet is one that we have already constructed. It arises from gauge theory on a half-space $\mathbb{R}^{1,2} \times L$ with the boundary condition constructed in Sect. 2.2.1 in which the gauge symmetry is reduced from G to H on the boundary.

Moreover, it should be clear now that this system can be coupled to any boundary hypermultiplets that parametrize a hyper-Kahler manifold Z with H action. The group $\hat{G}_H$ has a homomorphism to H by mapping a function $g(y)$ representing an element of $\hat{G}_H$ to its boundary value $g(0)$. So Z can be regarded as a hyper-Kahler manifold with $\hat{G}_H$ symmetry. Hence, we can simply borrow the standard formulas for coupling vector multiplets and hypermultiplets in three dimensions.

Equation (2.33) still holds and shows that in the presence of the boundary hypermultiplet, the boundary condition on $\vec{X}$ becomes

$$\vec{X}^{+}(0) + \vec{\mu}^{Z} = 0, \tag{2.34}$$

where $\vec{X}^{+}(0)$ is the projection of $\vec{X}(0)$ to $\mathfrak{h}$.

2.3.4 Coupling to a More General Boundary Theory

Hopefully, we have given a fairly clear recipe for coupling $\mathcal{N}=4$ super Yang-Mills in bulk to boundary hypermultiplets. One can also, without any difficulty, add vector multiplets that are supported on the boundary and couple to the same hypermultiplets. One simply replaces the group $\hat{G}$ in the above by $\hat{G} \times J$, where J is a finite-dimensional compact gauge group that "lives" at $y=0$. The boundary hypermultiplets can then be coupled to J as well as $\hat{G}$. Therefore, this recipe extends to the coupling of the bulk theory to any boundary theory of hypermultiplets and vector multiplets. The recipe is also useful for understanding the coupling to a more general CFT if that theory arises by renormalization group flow from a weakly coupled theory of vector multiplets and hypermultiplets with G action. Many interesting three-dimensional CFT's arise in this way.

To understand the coupling of $\mathcal{N}=4$ super Yang-Mills theory in bulk to a completely general CFT would require a more abstract approach that we will not develop here. One simple comment is that if this CFT has a Higgs branch, the description we have given is valid for describing the low energy coupling of the bulk $\mathcal{N}=4$ theory to that Higgs branch. (A full understanding that is not just valid at low energy would require returning to the underlying CFT.) Another useful point is that (2.33) holds in general, provided $\vec{\mu}^{Z}$ is understood as a suitable CFT operator (whose expectation value on the Higgs branch coincides with the classical hyper-Kahler moment map).

Going back to the simple case that Z parametrizes $\mathbb{R}^{4n}$ with a linear action of G, we would like to know that the coupling is conformally invariant quantum mechanically and not just classically. For a detailed treatment of a similar problem (involving bulk rather than boundary impurities), see [6]. A partial shortcut is to observe that global $\mathcal{N}=4$ supersymmetry in this situation actually implies superconformal symmetry. A collection of free hypermultiplets supported on a hyperplane or a boundary (and coupled to gauge fields in bulk) simply does not admit any possible counterterm of scaling dimension 3 or less that preserves global $\mathcal{N}=4$ supersymmetry.

2.3.5 Shifting the Boundary Conditions

Finally, we want to describe from the present point of view the possibility, explained in Sect. 2.2.3, to shift the boundary conditions on $\vec{X}$ and $\vec{Y}$ by constants.

In general, in coupling a vector multiplet to hypermultiplets, one is free to add a constant to the moment map, as long as this preserves gauge invariance. The resulting parameters are usually called Fayet-Iliopoulos (FI) parameters. In the present context, this means that we can pick any triple $\vec{v}$ valued in the center of $\mathfrak{h}$, and shift the moment map by a boundary term proportional to $\vec{v}$. Equation (2.32) then becomes

$$\vec{\mu}(y) = \frac{D\vec{X}}{Dy} + \vec{X} \times \vec{X}(y) + \delta(y)(\vec{X}(0) + \vec{\mu}^Z - \vec{v}), \tag{2.35}$$

and the boundary condition (2.34) on $\vec{X}$ becomes

$$\vec{X}^+(0) + \vec{\mu}^Z = \vec{v}. \tag{2.36}$$

This is the boundary condition of Sect. 2.2.3, or more precisely the generalization of it to include the coupling to a boundary matter system with moment map $\vec{\mu}^Z$.

Now let us discuss the other term in (2.25), the shift in the boundary value of $\vec{Y}^-$ by elements $\vec{w} \in \mathfrak{g}^-$ that commute with each other and with $\mathfrak{h}$. As they commute with H, the components of $\vec{w}$ are elements of the Lie algebra of the global symmetry group K described in Sect. 2.2.2. As they commute with each other, the components of $\vec{w}$ can be conjugated to a maximal torus T_K of K. Thus, they lie in an abelian group of global symmetries.

In three-dimensional $\mathcal{N} = 4$ supersymmetry with a finite dimensional gauge group coupled to hypermultiplets, an abelian group F of global symmetries leads to parameters—often called mass terms—that can be incorporated in the theory. The standard way to describe these parameters is to weakly gauge F, give expectation values to the scalar fields in the vector multiplet of F, and then turn off the gauge coupling of F.

It is not clear to us whether, in our situation with an infinite-dimensional gauge group, one can introduce the mass parameters in precisely this way.[6] We therefore offer the following alternative for introducing the mass parameters $\vec{w}$ in our situation.

We recall first that the Lie algebra of $\hat{G}_H$ consists of functions $\phi : L \to \mathfrak{g}$ such that $\phi(0) \in \mathfrak{h}$, or equivalently $\phi^-(0) = 0$. For any element $c \in \mathcal{Z}(\mathfrak{g}^-)$ (the subspace of $\mathfrak{g}^-$ that commutes with $\mathfrak{h} = \mathfrak{g}^+$), we can deform the adjoint representation of $\mathfrak{g}$ to the space of functions $\phi : L \to \mathfrak{g}$ that obey $\phi^-(0) = c$. Such a continuous deformation of a representation has no analog for a finite-dimensional compact group.

Now we modify the $\hat{G}_H$ vector multiplet as follows. We make no change in the three-dimensional $\hat{G}_H$ gauge fields A_μ, or in the fermions. But instead of interpreting $\vec{Y}$ as three scalar fields valued in the adjoint representation of $\hat{G}_H$, and thus obeying the boundary condition $\vec{Y}^-(0) = 0$, we consider each component Y_m, $m = 1, 2, 3$ to take values in a deformed adjoint representation with $c = w_m$.

Though the fields Y_m are not quite adjoint-valued, their commutators with each other or with adjoint-valued fields such as the other fields in the vector multiplet are adjoint-valued. And their commutators with hypermultiplet fields take values in the same spaces as

[6] One can gauge the global symmetry T_K, which means the following. Let $H' = H \times T_K$. Then repeating the analysis of Sect. 2.3.3 with H' replacing H, we arrive at a theory in which T_K has been gauged. But it does not seem to be natural to vary the T_K gauge coupling independently of the bulk G gauge coupling. This problem has no analog for finite-dimensional gauge groups.

at $w_m = 0$. To verify these statements, one uses the fact that the w_m commute with each other and with H, so that their presence does not affect the relevant properties of commutators. Given these facts, the three-dimensional supersymmetric action with gauge group $\hat{G}_H$ can be defined, and supersymmetry verified, in the usual way, despite the deformation of the adjoint representation.

2.4 The D5 System and Nahm's Equations

A vector multiplet with Neumann boundary conditions can be coupled to boundary degrees of freedom, as described in Sect. 2.3. What can be the dual of this for a vector multiplet with Dirichlet boundary conditions? This question may seem puzzling, because if a gauge field is required to vanish on the boundary, there is no obviously natural way to couple it to boundary degrees of freedom. The answer to this question turns out to be that half-BPS boundary conditions with Dirichlet boundary conditions on gauge fields are automatically coupled, in effect, to certain boundary degrees of freedom.

$\mathcal{N}=4$ super Yang-Mills theory on $\mathbb{R}^{1,3}$ has supersymmetric vacua parametrized by expectation values of $\vec{X}$ and $\vec{Y}$. To ensure supersymmetry, these expectation values must all commute. What happens on a half-space? It no longer makes sense, of course, to look for vacua with unbroken *four*-dimensional Poincaré supersymmetry, but we can look for vacua with *three*-dimensional Poincaré supersymmetry. Three-dimensional Poincaré invariance requires that $F_{\mu\nu}$ and $F_{3\mu}$ should vanish. It allows $\vec{X}$ and $\vec{Y}$ to have expectation values, depending only on y. We want to determine the condition on $\vec{X}(y)$ and $\vec{Y}(y)$ that ensures supersymmetry.

The supersymmetry variation of the fermion fields Ψ of $\mathcal{N}=4$ super Yang-Mills theory is conveniently written

$$\delta\overline{\Psi} = \frac{1}{2}\overline{\varepsilon}\Gamma^{IJ}F_{IJ}. \tag{2.37}$$

The condition for supersymmetry is simply that the right hand side must vanish:

$$\overline{\varepsilon}\Gamma^{IJ}F_{IJ} = 0. \tag{2.38}$$

This is the same as the condition (2.9) for a supersymmetric boundary condition, except that the factor $\Gamma_3\Psi$ is missing. Consequently, the equations resulting from (2.37) are the same as (2.12) that characterize supersymmetric boundary conditions, with the very important difference that the factor of ϑ should be omitted—so that in effect we must satisfy (2.12) for all choices of ϑ.

After imposing three-dimensional Poincaré invariance, we are left with three equations:

$$\begin{aligned} 0 &= [X_a, Y_m]\cdot\overline{\varepsilon}_0 B_0,\\ 0 &= \overline{\varepsilon}_0\left([X_b, X_c] - \epsilon_{abc}D_3X_aB_1\right),\\ 0 &= \overline{\varepsilon}_0\left([Y_m, Y_n] - \epsilon_{pmn}D_3Y_pB_2\right). \end{aligned} \tag{2.39}$$

The first tells us that all components of $\vec{X}$ and $\vec{Y}$ commute. The second tells us that unless $\overline{\varepsilon}_0$ is an eigenvector of B_1, we have $D\vec{X}/Dy = [\vec{X}, \vec{X}] = 0$. As a result, $\vec{X}$ coincides everywhere with its value at spatial infinity (up to a gauge transformation), and the different components of $\vec{X}$ must commute. The third equation similarly tells us that unless $\overline{\varepsilon}_0$ is an eigenvector of B_2, $\vec{Y}$ is a commuting constant and coincides with its value at spatial infinity. Thus, for generic $\overline{\varepsilon}_0$, all components of $\vec{X}$ and $\vec{Y}$ commute with each other and are covariantly constant.

Something interesting happens only if $\overline{\varepsilon}_0$ is an eigenvector of B_1 or B_2. We will take $\overline{\varepsilon}_0$ to be an eigenvector of B_1. (As usual, the case that $\overline{\varepsilon}_0$ is an eigenvector of B_2 simply differs by exchanging $\vec{X}$ and $\vec{Y}$.) If $\overline{\varepsilon}_0 B_1 = \pm\overline{\varepsilon}_0$, then the condition for supersymmetry gives

$$\frac{DX^1}{Dy} = \pm[X^2, X^3], \tag{2.40}$$

and cyclic permutations. It also implies that $\vec{Y}$ is a covariant constant whose components commute with each other and with $\vec{X}$:

$$\frac{D\vec{Y}}{Dy} = [\vec{Y}, \vec{Y}] = [\vec{Y}, \vec{X}] = 0. \tag{2.41}$$

More briefly, the components of $\vec{Y}$ generate unbroken gauge symmetries.

Equations (2.40) are known as Nahm's equations [1], and arise frequently as conditions for supersymmetry. Even after specifying the behavior of $\vec{X}$ at infinity, Nahm's equations have an interesting moduli space of solutions, which we will explore in Sect. 3. The existence of this moduli space means that, when vector multiplets obey Dirichlet boundary conditions, as happens in the D5-like case, there are in a sense boundary degrees of freedom already present in the theory. The dual of this for gauge fields with Neumann boundary conditions is that in that case, boundary degrees of freedom can be naturally added, as in Sect. 2.3.

2.4.1 Poles

Nahm's equations have another important consequence. Poles in the solutions of Nahm's equations can be used to generate new half-BPS boundary conditions. Though it may sound exotic, this idea is not new; it reflects the familiar fact [3, 12] that D3-branes ending on D5-branes can be described by solutions of Nahm's equations with poles. For related reasons, such poles played a crucial role in Nahm's original use of his equation [1]. Defining a new boundary condition by requiring a pole of a specified type is somewhat analogous to defining 't Hooft operators in gauge theory (or disorder operators in statistical mechanics) by requiring a singularity of a prescribed type.

The basic singular solution of Nahm's equation is simple to describe. With one choice of sign, Nahm's equations can be written

$$\frac{\mathrm{d}X^1}{\mathrm{d}y} + [X^2, X^3] = 0, \tag{2.42}$$

and cyclic permutations. Now let t^1, t^2, t^3 be any elements of $\mathfrak{g}$ that obey the $\mathfrak{su}(2)$ commutation relations $[t^1, t^2] = t^3$, and cyclic permutations. Thus, specifying the t^i amounts to specifying a homomorphism of Lie algebras $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. Having made such a choice, we obtain a solution of Nahm's equations with a pole at the origin:

$$X^i(y) = \frac{t^i}{y}. \tag{2.43}$$

So far, when we have discussed gauge theory on the half-space $y \geq 0$, we have considered fields that are regular on this half-space, including its boundary at $y = 0$, and the question has been what types of boundary values are allowed. Somewhat as in the definition of 't Hooft operators, we can introduce a new type of boundary condition by requiring a singularity of a prescribed type at $y=0$. If we wish in this way to get a supersymmetric boundary condition, we must select a singularity that is compatible with supersymmetry. The singularity $X^i \sim t^i/y$ clearly has this property, since it is compatible with Nahm's equations.

So for every choice of a non-zero homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, we get a new half-BPS boundary condition as follows. Setting t^i to be the images of a standard set of $\mathfrak{su}(2)$ generators, we require that the behavior of X^i near $y=0$ is $X^i \sim t^i/y$.

This preserves the same supersymmetry that is preserved by Dirichlet boundary conditions on gauge fields, since that is the supersymmetry that is preserved by Nahm's equations. A boundary condition of this type breaks the gauge symmetry from G to the subgroup G' that commutes with ρ. This gives a different type of half-BPS boundary condition with reduced gauge symmetry from what was described in Sect. 2.2.1. For the same reason as in that case, there is a group of global symmetries. This group is F, the commutant of ρ in G (that is, the subgroup of G that commutes with ρ).

As we explain next, the two constructions can be combined, roughly speaking by gauging a subgroup of F.

2.5 Combining the Constructions

We have described a significant generalization of each of the most obvious half-BPS boundary conditions. Neumann boundary conditions were generalized in Sect. 2.3 by including a boundary CFT. Dirichlet boundary conditions were generalized in Sect. 2.4 using a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. And orbifold boundary conditions were generalized in Sect. 2.2.1 to depend on a choice of an arbitrary subgroup H of the gauge group. It is possible to combine all three constructions, preserving the same supersymmetry, which we take to be of D5-type.

We will make the construction in three steps. Choosing an $\mathfrak{su}(2)$ embedding ρ, we require that $\vec{X}$ should have the familiar pole $\vec{X} \sim \vec{t}/y$.

Fields that do not commute with ρ will all vanish at the boundary, because of terms in the Hamiltonian that involve commutators with $\vec{X}$. Denoting therefore as $\boldsymbol{f}$ the Lie algebra of F (the commutant of ρ), what remains is to describe supersymmetric boundary conditions for the $\boldsymbol{f}$-valued parts of all fields. For this, in brief, we can use any supersymmetric boundary condition in F gauge theory. We pick any subgroup H of F and decompose $\boldsymbol{f} = \boldsymbol{f}^+ \oplus \boldsymbol{f}^-$, where $\boldsymbol{f}^+ = \mathfrak{h}$ and $\boldsymbol{f}^-$ is the orthocomplement. Then as in Sect. 2.2.1, we expand any field Φ as $\Phi^+ + \Phi^-$, with $\Phi^\pm \in \boldsymbol{f}^\pm$. We impose the boundary conditions described in Sect. 2.2.1:

$$\begin{aligned} F^+_{3\mu}| = F^-_{\mu\nu}| &= 0, \\ D_3 X^-| = X^+| &= 0, \\ Y^-| = D_3 Y^+| &= 0. \end{aligned} \tag{2.44}$$

The condition $F^-_{\mu\nu} = 0$ means that the curvature restricted to the boundary is $\mathfrak{h}$-valued, so that the gauge group along the boundary is H.

If we take H to be trivial, so that for every field Φ, $\Phi^+ = 0$ and $\Phi = \Phi^-$, this reduces to the boundary condition of Sect. 2.4.1. Whatever H may be, since the gauge symmetry along the boundary is H, we can introduce boundary hypermultiplets (or more general boundary variables) with H symmetry and couple them to the bulk gauge fields. When we do this, the boundary condition on $\vec{X}$ shifts from $\vec{X}^+| = 0$ to $\vec{X}^+| + \vec{\mu}^Z = 0$, where $\vec{\mu}^Z$ is the moment map for the boundary variables.

A unified way to describe the whole construction is to follow the logic of Sect. 2.3. We construct a three-dimensional supersymmetric gauge theory with an infinite-dimensional gauge group $\hat{G}_H$ consisting of maps $g : L \to G$ such that $g(0) \in H$. The bulk vector multiplets are $(A_\mu, \vec{Y})$. We couple to hypermultiplets $(\vec{X}, A_3)$ that are adjoint-valued but such that $\vec{X}$ is required to have the pole $\vec{X} \sim \vec{t}/y$ determined by ρ. We add additional boundary hypermultiplets (and possibly vector multiplets) as desired. The supersymmetric action we want then arises from the standard construction of a three-dimensional supersymmetric gauge theory with vector multiplets and hypermultiplets.

At this stage, we can follow the logic of Sect. 2.3.5 and introduce some additional parameters. These parameters are a triple $\vec{v}$ of elements of the center of $\mathfrak{h}$, and a triple $\vec{w}$ of elements of $\boldsymbol{f}^-$ that commute with each other and with $\mathfrak{h}$. The parameters are introduced by shifting the boundary conditions, which become

$$\begin{aligned} \vec{X}^+| + \vec{\mu}^Z &= \vec{v}, \\ \vec{Y}^-| &= \vec{w}. \end{aligned} \tag{2.45}$$

The general maximally supersymmetric boundary condition that we know of[7] thus involves a triple (ρ, H, B), where ρ is a homomorphism from $\mathfrak{su}(2)$ to $\mathfrak{g}$, H is a subgroup of G that commutes with ρ, and B is an $\mathcal{N} = 4$ supersymmetric field theory with H symmetry. The parameters that such a boundary condition depends upon (after fixing the parameters of the bulk theory) are a triple $\vec{v}$ in the center of $\mathfrak{h}$, a triple $\vec{w} \in \boldsymbol{f}^-$ whose components commute with each other and with $\mathfrak{h}$, and the parameters of the theory B.

2.5.1 A Brane Construction

Since this general construction may seem rather elaborate, we illustrate it with a brane configuration (Fig. 1). However, the reader may find the description of this brane configuration clearer after reading Sect. 3.

In the figure, we consider a $U(n)$ gauge theory associated to n parallel D3-branes, whose worldvolumes extend in directions 0123. These D3-branes extend to infinity in $y = x^3$ in one direction. They terminate in the other direction on D5-branes that extend in the 012456 directions and NS5-branes that extend in the 012789 directions. Reading the figure from right to left, first several D3-branes end on the same D5-brane. This gives a pole in Nahm's equations with a non-trivial embedding $\rho : \mathfrak{su}(2) \to \mathfrak{u}(n)$. Then, several D3-branes end one each on its own D5-branes. This gives a subalgebra of $\mathfrak{u}(n)$ in which $\vec{X}$ (which represents motion in the 456 directions) obeys Neumann boundary conditions and $\vec{Y}$ (which represents motion in the 789 directions) obeys Dirichlet boundary conditions. Finally, several D3-branes end on an NS5-brane, giving a subalgebra of $\mathfrak{u}(n)$ in which $\vec{X}$ obeys Dirichlet boundary conditions and $\vec{Y}$ obeys Neumann boundary conditions.

The figure is drawn for $n = 7$, so the gauge group is $G = U(7)$. The embedding $\rho : \mathfrak{su}(2) \to \mathfrak{u}(7)$ is of rank 3 and reduces the gauge symmetry to $F = U(4) \times U(1)$, and as the number of D3-branes ending on the NS5-brane is 2, the group H that remains as a gauge group at the boundary is $H = U(2)$. If the number of NS5-branes is greater than 1, the H gauge theory is coupled to a non-trivial boundary conformal field theory.

[7] Some of these boundary conditions can be generalized to include the θ angle [10]. The unbroken supersymmetry is then not of D5-type, but rotated by an outer automorphism of $PSU(4|4)$.

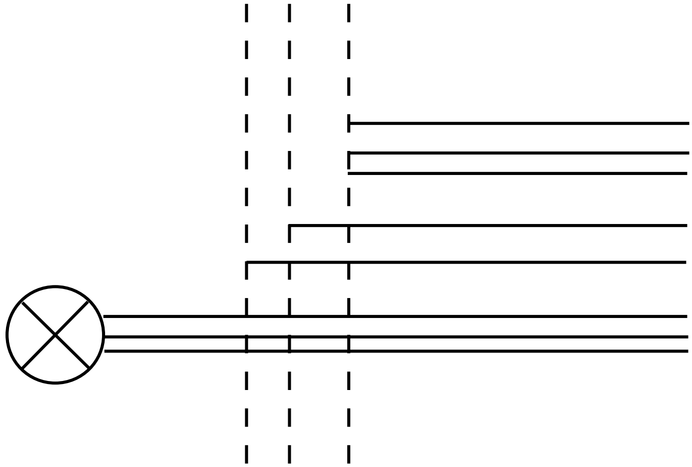

Fig. 1 A brane configuration whose purpose is to illustrate the general half-BPS boundary condition. A collection of semi-infinite D3-branes with worldvolume in the 0123 directions (portrayed by *horizontal solid lines*) ends on a collection of D5-branes that run in the 012456 directions (portrayed by *vertical dotted lines*) and one or more coincident NS5-branes that run in the 012789 directions (portrayed by the symbol $\otimes$). In this and subsequent pictures, the horizontal direction parametrizes x^3 and the vertical direction represents the 456 directions in spacetime

The parameters $\vec{v}$ by which the boundary conditions on $\vec{X}$ can be shifted arise from displacing the NS5-brane (or branes) in the 456 directions. The parameters $\vec{w}$ by which the boundary conditions on $\vec{Y}$ can be shifted arise from displacing the D5-branes in the 789 directions.

In the figure, to make the physics easier to describe, the various fivebranes have been displaced from each other in the y direction. To reduce to the case of gauge theory on a half-space with a boundary condition, one must take the limit in which all fivebranes become coincident in y.

This example thus illustrates all of the ideas that are used in constructing boundary conditions.

2.6 Domain Walls

A close cousin of the problem of supersymmetric boundary conditions is the problem of supersymmetric domain walls. The theory of half-BPS domain walls in $\mathcal{N}=4$ super Yang-Mills theory is known to be quite rich; many examples have been constructed in the string theory literature.

In fact, we do not really need anything new to describe such domain walls in field theory, since the problem of domain walls can be reduced to the problem that we have already considered of boundary conditions. Suppose that we want $\mathcal{N}=4$ super Yang-Mills theory with one gauge group G_1 in the half-space $x^3<0$, and another gauge group G_2 in the half-space $x^3>0$. What sort of half-BPS domain walls can interpolate between these two theories?

We can reduce this question to one that we have already studied by a simple "folding" trick. Instead of saying that there is one gauge theory to the left of the domain wall and one to the right, we can flip the "left" theory over to the right and say that the theory is trivial for $x^3<0$, and has gauge group $G_1\times G_2$ for $x^3>0$.

In folding or unfolding, we also must reverse the sign of three of the scalar fields in the gauge theory factor that is flipped between $x^3 > 0$ and $x^3 < 0$; merely changing the sign of x^3 is not a symmetry of the theory. To preserve D5-type supersymmetry, we should reverse the sign of $\vec{X}$.

So the problem of finding a domain wall that interpolates between G_1 and G_2 is equivalent to describing boundary conditions in the theory with gauge group $G_1 \times G_2$. For this, we can use any of the constructions that we have seen above, all of which are applicable to a general compact gauge group, not necessarily simple.

2.6.1 First Example

Let us give a few illustrative examples, in which we assume that ε_0 is an eigenvector of B_1 or B_2. Take $G_1 = G_2 = G$, so that the gauge group away from the boundary is $G \times G$. Let H be a copy of G diagonally embedded in $G \times G$. As in Sect. 2.2.1, we can find in $G \times G$ gauge theory a half-BPS boundary condition that breaks the $G \times G$ gauge symmetry in bulk down to H on the boundary. In fact, we do not really need the arguments of Sect. 2.2.1 for this particular example; since we have taken $G_1 = G_2 = G$, the unfolded theory simply has gauge group G everywhere and is ordinary $\mathcal{N} = 4$ super Yang-Mills with that gauge group. (One can verify that the arguments of Sect. 2.2.1 give the same result as "folding" $\mathcal{N} = 4$ super Yang-Mills to a theory with gauge group $G \times G$ on a half-space.) According to Sect. 2.3.3, we can furthermore modify the folded theory by coupling to boundary hypermultiplets that parametrize any hyper-Kahler manifold Z with H action. In the unfolded theory, what we have done is to couple $\mathcal{N} = 4$ super Yang-Mills theory with gauge group G to hypermultiplets that are supported on the hyperplane $x^3 = 0$. An example coming from the D3-D5 system has been treated in detail in [6].

2.6.2 Generalization

To generalize this, take any group G and subgroup G', with an embedding $i : G' \to G$. Let H be a copy of G', regarded as a subgroup of $G \times G'$ via the diagonal embedding $i \times 1 : H \to G \times G'$. Consider $G \times G'$ gauge theory on a half-space, with the half-BPS boundary conditions constructed in Sect. 2.2.1 that break $G \times G'$ down to H on the boundary. In the unfolded theory, this corresponds to a supersymmetric domain wall with gauge group G' on one side and G on the other. Various examples have been constructed in string theory via branes and fluxes. The model can be modified to include hypermultiplets with an arbitrary action of H supported on the domain wall.

This example can also be generalized to allow $\vec{X}$ to have a pole at $y = 0$, along the lines of (2.43). (The pole is in $\vec{X}$ rather than $\vec{Y}$ because of our choice of the unbroken supersymmetry.)

In this example, it is not necessary to assume that G' is a subgroup of G. We can take an arbitrary pair of gauge groups G and G', and a third group H with two embeddings $i : H \to G$ and $i' : H \to G'$. We regard H as a subgroup of $G \times G'$ via the diagonal embedding $i \times i' : H \to G \times G'$, and consider a half-BPS boundary condition with $G \times G'$ gauge symmetry in a half-space reduced to H on the boundary, possibly coupled to boundary hypermultiplets with H action. In the unfolded theory, this sort of construction gives half-BPS domain walls interpolating between gauge group G on one side and G' on the other. The subgroup of $G \times G'$ that commutes with H acts as global symmetries at the boundary.

3 Moduli Spaces of Solutions of Nahm's Equations

As we explained in Sect. 2.4, in $\mathcal{N}=4$ super Yang-Mills theory on a half-space with suitable D5-like boundary conditions, supersymmetric vacua arise from solutions of Nahm's equations

$$\frac{\mathrm{d}X_i}{\mathrm{d}y} + [X_{i+1}, X_{i-1}] = 0, \quad i = 1, 2, 3 \tag{3.1}$$

on the half-line $L : y \geq 0$. $\vec{X}$ must also commute with the constant value of $\vec{Y}$ at $y = \infty$; until Sect. 3.7, we assume that this constant value vanishes.

What we really want to define is the moduli space of vacua of the half-space theory for a given choice of the vacuum at infinity. The vacuum at infinity is specified by a choice (up to conjugation by a constant gauge transformation) of the value of $\vec{X}$ at $y = \infty$. We write $\vec{X}_\infty = (X_{1,\infty}, X_{2,\infty}, X_{3,\infty})$ for this limiting value; the components of $\vec{X}_\infty$ must commute. It is convenient to first consider the case that $\vec{X}_\infty$ is regular, in the sense that the subgroup of G that commutes with all components of $\vec{X}_\infty$ is precisely a maximal torus T. We let $\mathcal{M}$ denote the moduli space of solutions of Nahm's equations with some appropriate condition at $y = 0$, and with $\vec{X}(y) \to \vec{X}_\infty$ (up to conjugation) for $y \to \infty$.

For the relevant boundary conditions, $\vec{X}$ is part of a hypermultiplet, and therefore it is natural to think of $\mathcal{M}$ as a Higgs branch of vacua. On general grounds, $\mathcal{M}$ is a hyper-Kahler manifold. In fact, the relevant spaces of solutions of Nahm's equations were used by Kronheimer [13, 14] to define hyper-Kahler metrics on certain spaces that arise in representation theory. For reviews and some later refinements, see [15, 16]. We will try to give a fairly self-contained explanation of the facts we need about Nahm's equations, but essentially everything we explain is contained in the above-cited references, or in the literature on Nahm's equations applied to BPS monopoles in three dimensions (where those equations originally arose [1]). For a recent survey of the extensive literature on Nahm's equations and monopoles, see [2]. For previous results from a D-brane perspective, see [3–5].

3.1 The Hyper-Kahler Quotient

The proof that $\mathcal{M}$ is hyper-Kahler (see [14], Sect. 3) uses the fact that it can be interpreted as a hyper-Kahler quotient. We follow the logic of Sect. 2.4. We complete $\vec{X}$ to a hypermultiplet by adding $A = A_3$, the component of the gauge field in the y direction. We pick a maximal torus T with Lie algebra $\mathfrak{t}$, and we pick a regular triple $\vec{X}_\infty \in \mathfrak{t}$. We require that $\vec{X} \to \vec{X}_\infty$ for $y \to \infty$. (For the moment, we place no restriction on $\vec{X}(0)$ except that it should be non-singular.) And we require that A is $\mathfrak{t}$-valued at infinity. $\vec{X}$ and A together parametrize a flat hyper-Kahler manifold $\mathcal{W}$. The three symplectic forms of $\mathcal{W}$ are

$$\omega_i = \int_L \mathrm{d}y\, \mathrm{Tr}(\delta A \wedge \delta X_i + \delta X_{i+1} \wedge \delta X_{i-1}), \quad i = 1, 2, 3. \tag{3.2}$$

We let $\hat{G}$ be the group of gauge transformations $g : L \to G$ such that $g(0) = 1$, and g is T-valued for $y \to \infty$. $\hat{G}$ acts on $\mathcal{W}$ with a hyper-Kahler moment map

$$\mu_i = \frac{DX_i}{Dy} + [X_{i+1}, X_{i-1}], \quad i = 1, 2, 3, \tag{3.3}$$

as in (2.31). (Because $g(0) = 1$, there is no delta function in the moment map at $y = 0$.) On general grounds, the hyper-Kahler quotient of $\mathcal{W}$ by $\hat{G}$ is a hyper-Kahler manifold $\mathcal{M}$. The hyper-Kahler quotient is obtained by setting to zero the moment map and dividing by $\hat{G}$.

A convenient way to describe $\mathcal{M}$ is to eliminate A. There is always a unique map $g : L \to G$, with $g(0) = 1$, such that a gauge transformation by g sets A to zero. After setting $A = 0$, the condition $\vec{\mu} = 0$ becomes Nahm's equations. However, g is not necessarily an element of $\hat{G}$, since it may not be T-valued for $y \to \infty$. So after eliminating A, we can no longer claim that $\vec{X}(y) \to \vec{X}_\infty$ for $y \to \infty$. Rather, $\vec{X}(y)$ approaches a limit for $y \to \infty$ and this limit is conjugate to $\vec{X}_\infty$ by a constant gauge transformation.

The hyper-Kahler manifold obtained this way depends on $\vec{X}_\infty$, of course, so we sometimes denote it as $\mathcal{M}(\vec{X}_\infty)$. $\mathcal{M}(\vec{X}_\infty)$ is smooth as long as $\vec{X}_\infty$ is regular (which is needed for the above construction to make sense as stated) because the condition that $g(0) = 1$ ensures that the gauge group acts freely on $\mathcal{W}$. Smoothness of $\mathcal{M}(\vec{X}_\infty)$ for regular $\vec{X}_\infty$ will also be clear in Sect. 3.2 when we describe $\mathcal{M}$ as a complex manifold. $\mathcal{M}(\vec{X}_\infty)$ can be continued to non-regular values—for instance, $\vec{X}_\infty = 0$—but as will also be clear in Sect. 3.2, it then develops singularities.

The original finite-dimensional group G acts on $\mathcal{M}(\vec{X}_\infty)$, by gauge transformations at $y = 0$. To compute the moment map for the G action, we just repeat the computation of (2.31), and then define $\vec{\mu} = \int \mathrm{d}y \vec{\mu}(y)$. Now we pick up a delta function contribution at $y = 0$, since we do not require $g(0) = 1$; indeed, since we are imposing Nahm's equations, the delta function is all we get. So the hyper-Kahler moment map for the G action is

$$\vec{\mu} = \vec{X}(0). \tag{3.4}$$

3.1.1 *Including a Pole*

As in Sect. 2.4.1, we can construct a more general boundary condition by choosing a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$ and requiring that for $y \to 0$

$$X_i(y) = \frac{t_i}{y} + \cdots . \tag{3.5}$$

Here t_i are the images under ρ of a standard basis of $\mathfrak{su}(2)$; the ellipses refer to terms regular at $y = 0$.

We denote the subgroup of G that commutes with ρ as H, and we call its Lie algebra $\mathfrak{h}$. We modify the above construction by requiring that A is $\mathfrak{h}$-valued at $y = 0$, so that $A(0)$ commutes with the polar part of $\vec{X}$.

The hyper-Kahler quotient now gives a hyper-Kahler manifold $\mathcal{M}_\rho(\vec{X})$ that (after gauging away A) parametrizes solutions of Nahm's equations with the behavior of (3.5) for $y \to 0$, and with $\vec{X} \to \vec{X}_\infty$ up to conjugation for $y \to \infty$.

$\mathcal{M}_\rho(\vec{X})$ admits an action of H. The hyper-Kahler moment map is

$$\vec{\mu} = \vec{X}_{\mathfrak{h}}(0). \tag{3.6}$$

Here $\vec{X}_{\mathfrak{h}}$ is the orthogonal projection of $\vec{X}$ from $\mathfrak{g}$ to $\mathfrak{h}$. Of course, $\vec{X}_{\mathfrak{h}}$ is regular at $y = 0$, even though $\vec{X}$ has a pole.

3.2 The Complex Manifold

For our purposes, the most useful way to understand the hyper-Kahler manifold $\mathcal{H}$ is to describe it as a complex manifold in one of its complex structures. (We continue to follow [14], Sect. 3.) We first consider the case that $\vec{X}$ has no pole at $y = 0$.

We let $\mathcal{X} = X_1 + iX_2$, $\mathcal{A} = A + iX_3$. In one complex structure on the infinite-dimensional hyper-Kahler manifold $\mathcal{W}$, the fields $\mathcal{X}$ and $\mathcal{A}$ are complex coordinates. In this complex structure, two of Nahm's equations combine to a single holomorphic equation

$$\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}y} = 0. \tag{3.7}$$

Here $\mathcal{D}\mathcal{X}/\mathcal{D}y = \mathrm{d}\mathcal{X}/\mathrm{d}y + [\mathcal{A}, \mathcal{X}]$ is the covariant derivative of $\mathcal{X}$ with respect to the complex-valued connection $\mathcal{A}$. In solving (3.7), we require that $\mathcal{X}(y) \to \mathcal{X}_\infty$ for $y \to \infty$, where $\mathcal{X}_\infty = X_{1,\infty} + iX_{2,\infty}$.

Equation (3.7) is invariant under complex-valued gauge transformations, acting in the usual way $\mathcal{X} \to g\mathcal{X}g^{-1}$, $\mathcal{D} \to g\mathcal{D}g^{-1}$, where now $g(y) : L \to G_\mathbb{C}$ takes values in the complexification $G_\mathbb{C}$ of G. We also require that $g(0) = 1$, and that g for large y is valued in $T_\mathbb{C}$, the complexification of T; these conditions mean that $g(y)$ is an element of $\hat{G}_\mathbb{C}$, the complexification of the group $\hat{G}$ that was used in the construction of $\mathcal{M}$ as a hyper-Kahler quotient.

By a standard type of argument,[8] imposing the third Nahm equation and dividing by $\hat{G}$ is equivalent to simply dividing by $\hat{G}_\mathbb{C}$. But dividing by $\hat{G}_\mathbb{C}$ is a very simple operation. If we relax the requirement that $g(y)$ is $T_\mathbb{C}$-valued at infinity, then there is a unique $G_\mathbb{C}$-valued gauge transformation, with $g(0) = 1$, that sets $\mathcal{A} = 0$. Since $g(\infty)$ may not commute with $\mathcal{X}_\infty$, after we make this gauge transformation $\mathcal{X}(y)$ is conjugate for $y \to \infty$ to $\mathcal{X}_\infty$ but need not equal $\mathcal{X}_\infty$.

In the gauge $\mathcal{A} = 0$, the complex Nahm equation (3.7) reduces to $\mathrm{d}\mathcal{X}/\mathrm{d}y = 0$, telling us that $\mathcal{X}$ is a constant. The boundary condition at $y = 0$ (which just says that $\mathcal{X}$ is finite there) puts no restriction on the constant, and the boundary condition at infinity simply tells us that $\mathcal{X}$ is conjugate to $\mathcal{X}_\infty$.

So as a complex manifold, $\mathcal{M}$ is isomorphic to the conjugacy class of $\mathcal{X}_\infty$ in the complex Lie algebra $\mathfrak{g}_\mathbb{C}$. In particular, this implies that if $\mathcal{X}_\infty$ is regular semi-simple (diagonalizable with distinct eigenvalues) then $\mathcal{M}$ is smooth. If $\vec{X}_\infty$ is regular, then $\mathcal{X}_\infty$ is regular semi-simple for a generic choice of coordinate axes (that is, a generic choice of which components of $\vec{X}$ we identify as $X_1 + iX_2$).

3.2.1 Conjugacy Classes in Complex Lie Algebras

Because of this result and related results that will soon appear, we need a few simple results on conjugacy classes in complex Lie groups.

Let G be a compact Lie group of dimension d, and let $G_\mathbb{C}$ be its complexification. Then the complex dimension of $G_\mathbb{C}$ is also d. Let x be an element of $\mathfrak{g}_\mathbb{C}$, and S the subgroup of $G_\mathbb{C}$ that commutes with x. Let s be the complex dimension of S. The orbit $\mathcal{O}_x$ of x in $\mathfrak{g}_\mathbb{C}$ is a complex manifold of complex dimension $d - s$.

The smallest possible value of s is r, the rank of G. For example, suppose that x can be conjugated to the Lie algebra $\mathfrak{t}_\mathbb{C}$ of a (complex) maximal torus $T_\mathbb{C}$. Then x at least commutes with $T_\mathbb{C}$, of dimension r, so $s \geq r$. If x is a generic element of $T_\mathbb{C}$, then $S = T_\mathbb{C}$ and $s = r$. x is said to be semisimple if it can be conjugated to a maximal torus, and regular if $s = r$.

The starting point in our analysis was an assumption that $\vec{X}_\infty$ is regular, meaning that the value of $\vec{X}$ at infinity breaks the gauge group G to its maximal torus T. ($\vec{X}_\infty$ is automatically

[8] Stability is not an issue if $\mathcal{X}_\infty$ is regular semi-simple, which for regular $\vec{X}$ is true for a generic choice of the coordinate axes in $\vec{X}$ space.

semisimple; indeed, supersymmetry requires that the components of $\vec{X}_\infty$ commute and so can be simultaneously conjugated to a maximal torus.) Then to avoid some technicalities we oriented the coordinate axes in a generic fashion, so that $\mathcal{X}_\infty = X_{1,\infty} + iX_{2,\infty}$ is also regular semi-simple. This means that the gauge symmetry breaking is fully reflected in $\mathcal{X}_\infty$.

3.2.2 Turning Off the Symmetry Breaking

It is also of interest to ask what happens when we turn off the gauge symmetry breaking at infinity. An important subtlety will arise, so to get our bearings we start with the example of $G = SU(2)$. If $\mathcal{X}_\infty$ is regular semi-simple, then it is conjugate to $\mathrm{diag}(w, -w)$ for some $w \in \mathbb{C}$. As a result, the quadratic Casimir invariant $u = \mathrm{Tr}\,\mathcal{X}^2$ is nonzero; in fact, $u = 2w^2$. u is a natural gauge-invariant measure of the symmetry breaking.

What happens if we take $u \to 0$? One might think that means that $\mathcal{X}$ goes to zero and its orbit collapses to a point. That is actually not the case. The following nonzero element of $\mathfrak{sl}(2,\mathbb{C})$ has $u = 0$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.8}$$

x is regular, since the subgroup S of $SL(2,\mathbb{C})$ that commutes with x is one-dimensional, being generated by x itself. In general, for every value of the Casimir invariants of a complex Lie group, there is a unique regular orbit. For $SL(2,\mathbb{C})$, u is the only independent Casimir orbit; the orbit x is the regular orbit with $u = 0$. For every u, a regular element w_u of $\mathfrak{sl}(2,\mathbb{C})$ with $\mathrm{Tr}\,w_u^2 = u$ can be written as follows:

$$w_u = \begin{pmatrix} 0 & 1 \\ u/2 & 0 \end{pmatrix}. \tag{3.9}$$

This family contains every regular conjugacy class precisely once.

The orbit $\mathcal{O}_x$ of $x = w_0$ can easily be described explicitly. Any element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.10}$$

of $\mathfrak{sl}(2,\mathbb{C})$ is conjugate to x if and only if

$$ad - bc = 0 \tag{3.11}$$

and a, b, c, d are not all zero.

Obviously, the orbit $\mathcal{O}_x$ is not closed in $\mathfrak{sl}(2,\mathbb{C})$. To take its closure, we must relax the condition that a, b, c, and d are not all zero. If we do relax this condition, we get a subspace of $\mathfrak{sl}(2,\mathbb{C})$ that is known as the nilpotent cone $\mathcal{N}$. It parametrizes all nilpotent elements of the Lie algebra, conjugate to x or not. For our example of $SL(2,\mathbb{C})$, $\mathcal{N}$ is the union of two orbits; one orbit is $\mathcal{O}_x$, and the second orbit is a single point, the orbit $\mathcal{O}_0$ of the zero element of $\mathfrak{sl}(2,\mathbb{C})$ with $a = b = c = d = 0$. In fact, $\mathcal{O}_0$ is a singularity of $\mathcal{N}$. The equation $ad - bc = 0$ that defines $\mathcal{N}$ is a standard description of the A_1 singularity. Topologically, for $SL(2,\mathbb{C})$, $\mathcal{N}$ is $\mathbb{C}^2/\mathbb{Z}_2$ or equivalently $\mathbb{R}^4/\mathbb{Z}_2$.

We can describe explicitly the family of solutions of the original real Nahm equations (3.1) that is parametrized by $\mathcal{N}$:

$$X_i(y) = g\frac{t_i}{y + f^{-1}}g^{-1}. \tag{3.12}$$

Here t_i are the standard 2×2 Pauli matrices, with $[t_1, t_2] = t_3$, etc., f is a non-negative real constant, and $g \in SU(2)$. For $f = 0$, $\vec{X}(y)$ identically vanishes; this is the trivial zero solution of Nahm's equations, which corresponds to the singular point in $\mathcal{N}$. For all $f \geq 0$, $\vec{X}(y)$ is regular on the whole half-line, including $y = 0$, and vanishes for $y \to \infty$.

It is not difficult to describe the topology of the manifold $\mathcal{M}$ that parametrizes this family of solutions of Nahm's equations. f takes values in the half-line $\mathbb{R}_{\geq 0}$, and (since g and $-g$ are equivalent in (3.12)) g takes values in $SU(2)/\mathbb{Z}_2 = S^3/\mathbb{Z}_2$. So $\mathcal{M} = S^3/\mathbb{Z}_2 \times \mathbb{R}_{\geq 0}$. But this is the same as $\mathbb{R}^4/\mathbb{Z}_2$, which is the same as $\mathbb{C}^2/\mathbb{Z}_2$ and coincides with the nilpotent cone $\mathcal{N}$ for $SL(2,\mathbb{C})$. In fact, one can readily verify that in this family of solutions, $\mathcal{X}(0) = X_1(0) + iX_2(0)$ is always nilpotent, and that every nilpotent element of $\mathfrak{sl}(2,\mathbb{C})$ equals $\mathcal{X}(0)$ for precisely one choice of g (up to sign) and f.

Going back to the original problem, for $G = SU(2)$, if $\vec{X}_\infty$ is regular, then the moduli space $\mathcal{M}$ of solutions of Nahm's equations is a smooth manifold that, in a generic complex structure, is the orbit of a regular semisimple element of $\mathfrak{sl}(2,\mathbb{C})$. But if we turn off the symmetry breaking and set $\vec{X}_\infty = 0$, then $\mathcal{M}$ becomes the nilpotent cone $\mathcal{N}$.

Starting from $\vec{X}_\infty = 0$, if we turn on $X_{1,\infty}$ and $X_{2,\infty}$, then $\mathcal{N}$ is deformed and becomes the smooth orbit of a regular semi-simple element $\mathcal{X}_\infty = X_{1,\infty} + iX_{2,\infty}$. But if we keep $X_{1,\infty} = X_{2,\infty} = 0$ and turn on $X_{3,\infty}$, then the singularity of $\mathcal{N}$ is resolved, rather than deformed.

3.2.3 Analog for Any G

The analog for any G is as follows. The complex Nahm equation

$$\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}y} = 0 \tag{3.13}$$

implies that the Casimir invariants of $\mathcal{X}$ are independent of y. The Casimir invariants of $\mathcal{X}(0)$ therefore coincide with those of $\mathcal{X}_\infty$. $\mathcal{X}(0)$ is gauge-invariant (since we only divide by gauge transformations that equal 1 at $y = 0$). Up to a complex gauge transformation, the complex Nahm equation has a unique solution for every choice of $\mathcal{X}(0)$ that has the same Casimir invariants as $\mathcal{X}_\infty$.

If $\mathcal{X}_\infty$ is regular, then any element of $\mathfrak{g}_\mathbb{C}$ with the same Casimir invariants is conjugate to $\mathcal{X}_\infty$. Hence the moduli space $\mathcal{H}$ of solutions of Nahm's equations is simply the orbit of $\mathcal{X}_\infty$ in $\mathfrak{g}_\mathbb{C}$. We denote this orbit as $\mathcal{O}_{\mathcal{X}_\infty}$, and we denote as $\overline{\mathcal{O}}_{\mathcal{X}_\infty}$ the space of all elements of $\mathfrak{g}_\mathbb{C}$ with the same Casimir invariants as $\mathcal{X}_\infty$. These two spaces coincide precisely if $\mathcal{X}_\infty$ is regular.

Even if $\mathcal{X}_\infty$ is not regular, it is always possible to find a regular element $x \in \mathfrak{g}_\mathbb{C}$ with the same Casimir invariants as $\mathcal{X}_\infty$. For instance, generalizing the example for $SL(2,\mathbb{C})$, for $G_\mathbb{C} = SL(n,\mathbb{C})$, if $\mathcal{X}_\infty = 0$, we can take x to be an $n \times n$ matrix with 1's just above the main diagonal and all other matrix elements zero:

$$x = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.14}$$

(The subgroup of $SL(n,\mathbb{C})$ that commutes with x is generated by $x, x^2, \dots, x^{n-1}$, and so has the same dimension as a maximal torus.) The moduli space $\mathcal{H} = \overline{\mathcal{O}}_{\mathcal{X}_\infty}$ of solutions of

Nahm's equations is always the *closure* of the orbit $\mathcal{O}_x$ of x in $\mathfrak{g}_{\mathbb{C}}$. The closure is obtained by adding to $\mathcal{O}_x$ the orbits of non-regular elements x' that have the same Casimir invariants as x. In our example with $SL(2,\mathbb{C})$, x was a regular nilpotent element and the only relevant non-regular x' was $x'=0$; in general, finitely many non-regular orbits appear. The dimension of a regular orbit is greater than that of any non-regular orbit (since a regular element, by definition, has a centralizer of the minimum dimension) and the non-regular orbits $\mathcal{O}_{x'}$ appear as singularities in $\mathcal{H}$, just as in our example.

Physically, an important special case is the case that symmetry breaking is absent at $y=\infty$. This means that $\vec{X}_\infty = 0 = \mathcal{X}_\infty$, and therefore the Casimir invariants of $\mathcal{X}_\infty$ all vanish. An element $\mathcal{X}(0)$ of $\mathfrak{g}_{\mathbb{C}}$ has vanishing Casimir invariants if and only if it is nilpotent. Therefore, in this situation, the moduli space $\mathcal{H}$ of solutions of Nahm's equations coincides with the nilpotent cone $\mathcal{N}$ consisting of all nilpotent elements of $\mathfrak{g}_{\mathbb{C}}$. These make up finitely many conjugacy classes.

The orbit $\mathcal{O}_x$ of a regular nilpotent element x is a dense open set in $\mathcal{N}$. $\mathcal{N}$ actually equals $\overline{\mathcal{O}}_x$, the closure of $\mathcal{O}_x$; $\mathcal{N}$ has singularities corresponding to non-regular nilpotent orbits. Symmetry breaking at infinity (by the choice of $\vec{X}_\infty$) causes these singularities to be deformed and resolved; if one chooses $\vec{X}_\infty$ to break G to its maximal torus, then the moduli space of vacua becomes smooth.

3.2.4 More on Nilpotent Orbits

As we have just seen, nilpotent orbits in complex Lie algebras are important in our subject. So we pause for a few words on these orbits.

A nilpotent element of $\mathfrak{sl}(n,\mathbb{C})$ is simply an $n\times n$ nilpotent matrix. Any $n\times n$ complex matrix can be conjugated to a Jordan canonical form. The Jordan canonical form of a nilpotent matrix x has all matrix elements vanishing except for 1's in some of the entries just above the main diagonal. For some decomposition $n=n_1+n_2+\cdots+n_k$, with positive integers n_i, where we can assume $n_1\geq n_2\geq\cdots\geq n_k$, x takes a block-diagonal form in which the diagonal blocks are regular nilpotent $n_p\times n_p$ matrices, $1\leq p\leq k$, each taking precisely the form in (3.14). The off-diagonal blocks vanish.

An alternative description is useful for generalizing to any group. Let $\rho:\mathfrak{su}(2)\to\mathfrak{g}_{\mathbb{C}}$ be any homomorphism, and as usual write t_1,t_2,t_3 for the images of standard generators of $\mathfrak{su}(2)$. The "raising" operator $t_+=t_1+it_2$ is then nilpotent. Conversely, according to the Jacobson-Morozov Theorem, every nilpotent element of a complex semi-simple Lie algebra arises in this way from some $\mathfrak{su}(2)$ embedding.

Let us verify this assertion in the case of $SL(n,\mathbb{C})$. The Lie algebra $\mathfrak{su}(2)$ has, up to isomorphism, one irreducible representation of each positive integer dimension $1,2,3,\ldots$. So up to isomorphism, the embedding $\rho:\mathfrak{su}(2)\to\mathfrak{sl}(n,\mathbb{C})$ is determined by a decomposition $n=n_1+n_2+\cdots+n_k$, with positive integers n_i that we can assume to be non-increasing. Moreover, in an irreducible p-dimensional representation of $\mathfrak{su}(2)$, the raising operator is a regular nilpotent element, conjugate to the $p\times p$ case of the matrix described in (3.14). So the two descriptions agree.

One advantage of the description by $\mathfrak{su}(2)$ embeddings is that it gives a convenient way to determine the dimension of an orbit. Let t_+ be a nilpotent element of $\mathfrak{g}_{\mathbb{C}}$ that is the raising operator for some $\mathfrak{su}(2)$ embedding ρ, and let $\mathcal{O}_\rho$ be its orbit. The dimension of $\mathcal{O}_\rho$ will be $d-s$, where d is the dimension of $G_{\mathbb{C}}$ and s is the dimension of the subgroup S that commutes with t_+. (All dimensions here are complex dimensions.) What is s? Let us decompose $\mathfrak{g}_{\mathbb{C}}$ in irreducible representations $\mathcal{T}_j$ of $\mathfrak{su}(2)$:

$$\mathfrak{g}_{\mathbb{C}}=\oplus_{j=1}^{s}\mathcal{T}_j. \tag{3.15}$$

Elements of $\mathfrak{g}_{\mathbb{C}}$ that commute with the raising operator t_+ are precisely the highest weight vectors in the summands $\mathcal{T}_j$. Each $\mathcal{T}_j$ has a one-dimensional space of highest weight vectors. Therefore the number s of summands in (3.15) is the dimension of the centralizer S of t_+. The dimension of the orbit of t_+ is therefore $d-s$.

Let us check this calculation for the case of a regular nilpotent element x. This is the case that $\rho:\mathfrak{su}(2)\to\mathfrak{sl}(n,\mathbb{C})$ is associated with an irreducible n-dimensional representation of $\mathfrak{su}(2)$. For this representation, the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ decomposes as a direct sum of $\mathfrak{su}(2)$ modules of dimensions $3,5,7,\dots,2n-1$. There are $n-1$ pieces in all, so $s=n-1$, as we computed before in another way.

Some More Examples It will be helpful to give a few more examples of nilpotent orbits.

As we have already explained, for every simple Lie group G, there is a unique regular nilpotent orbit. For $G=SU(n)$, it corresponds to an irreducible embedding $\rho:\mathfrak{su}(2)\to\mathfrak{su}(n)$. The regular nilpotent orbit is a dense open set in the nilpotent cone $\mathcal{N}$.

If G is simply-laced, there is also a unique subregular nilpotent orbit $\mathcal{O}'$—one whose complex dimension is precisely 2 less than the dimension of $\mathcal{N}$. $\mathcal{O}'$ therefore appears as a locus of singularities in $\mathcal{N}$, and (in keeping with the hyper-Kahler nature of $\mathcal{N}$) these are orbifold singularities $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$. In fact, Γ is the finite subgroup of $SU(2)$ that corresponds to G in the usual mapping between such subgroups and simple groups of type A-D-E.

For $G=SU(2)=\mathrm{A}_1$, the subregular nilpotent element is simply the zero element. The fact that the nilpotent cone $\mathcal{N}$ has an A_1 singularity corresponding to the zero element is a special case of the general relation of subregular nilpotent orbits to A-D-E singularities.

More generally, for $G=SU(n)$, the subregular nilpotent orbit is the raising operator t_+ of an $SU(2)$ embedding that corresponds to a decomposition $n=(n-1)+1$. A computation as above shows that the centralizer of such a t_+ has dimension $n+1$, which exceeds by 2 the rank $n-1$ of $SU(n)$. This accounts for the fact that the orbit $\mathcal{O}'$ is of codimension 2 in the nilpotent cone.

At the other extreme, the zero element of $\mathfrak{g}_{\mathbb{C}}$ is the unique nilpotent element whose orbit consists of a single point. There is also a unique nilpotent orbit of smallest positive dimension—usually called the minimal (non-zero) nilpotent orbit. For $G=SU(n)$, it corresponds to the decomposition $n=2+1+1+\cdots+1$. A computation as above shows that the corresponding orbit must have dimension $2n-2$. In fact, this orbit consists of $n\times n$ matrices M of rank 1 with $M^2=0$. Such a matrix can be written $M^i{}_j=B^iC_j$, where $\sum_i B^iC_i=0$; this way of writing M is unique modulo $B\to\lambda B$, $C\to\lambda^{-1}C$.

3.3 Solutions of Nahm's Equations with Poles

In Sect. 3.1.1, we considered Nahm's equations with a pole at $y=0$ determined by a homomorphism $\rho:\mathfrak{su}(2)\to\mathfrak{g}$. As we explained there, the solutions of Nahm's equations, with boundary conditions that $\vec{X}$ is conjugate at infinity to a commuting triple $\vec{X}_\infty$, are parametrized by a hyper-Kahler manifold $\mathcal{M}_\rho(\vec{X})$.

We proceed, following [13], just as in the case of trivial ρ. Setting $\mathcal{X}=X_1+iX_2$, $\mathcal{A}=A+iX_3$, two of Nahm's equations combine to a form familiar from (3.7):

$$\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}y}=0. \tag{3.16}$$

Now, however, $\mathcal{X}$ and $\mathcal{A}$ are not regular at $y=0$. Rather, we have

$$\mathcal{X}=\frac{t_1+it_2}{y}+\cdots=\frac{t_+}{y}+\cdots, \tag{3.17}$$

$$\mathcal{A} = \frac{it_3}{y} + \cdots, \tag{3.18}$$

where the ellipses denote regular terms.

As before, Nahm's three real equations modulo real gauge transformations are equivalent to the complex equation (3.16) modulo complex gauge transformations. Now, however, we cannot use a complex gauge transformation to set $\mathcal{A} = 0$. The reason for this is that we are restricted to gauge transformations that are trivial at $y = 0$. A gauge transformation that would remove the singularity from $\mathcal{A}$ would have to have a singularity at $y = 0$.

We can, however, make a gauge transformation to set $\mathcal{A} = it_3/y$ everywhere. Just as in our previous analysis, the gauge transformation that does this may not commute with $\mathcal{X}_\infty$ for $y \to \infty$. So after setting $\mathcal{A} = it_3/y$, we should require that $\mathcal{X}(y)$ is conjugate to $\mathcal{X}_\infty$ for $y \to \infty$, not that the two are equal.

After setting $\mathcal{A} = it_3/y$, it is straightforward to solve the complex Nahm equation. We pick a basis v_α of $\mathfrak{g}$ of vectors of definite weight

$$[it_3, v_\alpha] = m_\alpha v_\alpha, \tag{3.19}$$

where $m_\alpha \in \mathbb{Z}/2$. (For example, $[it_3, t_\pm] = \pm t_\pm$, so we can take $t_\pm$ for two of the v_α.) Then the complex Nahm equation has the general solution

$$\mathcal{X} = \sum_\alpha \epsilon_\alpha \frac{v_\alpha}{y^{m_\alpha}} \tag{3.20}$$

with coefficients ϵ_α. However, we want solutions in which the singular part at $y = 0$ is precisely t_+/y. So we must have

$$\mathcal{X} = \frac{t_+}{y} + \sum_{m_\alpha \le 0} \epsilon_\alpha v_\alpha y^{-m_\alpha}. \tag{3.21}$$

This is not the whole story, because the gauge transformation that sets $\mathcal{A} = it_3/y$ is not unique. This form is preserved by a further gauge transformation generated by

$$\phi = \sum_\alpha f_\alpha v_\alpha y^{-m_\alpha}, \tag{3.22}$$

with arbitrary coefficients f_α. However, since we are supposed to allow only gauge transformations that vanish at $y = 0$, we must actually restrict the coefficients so that $\phi = \sum_{m_\alpha < 0} f_\alpha v_\alpha y^{-m_\alpha}$. By a gauge transformation that shifts $\mathcal{X}$ by $[\phi, \mathcal{X}]$ with ϕ of this form, we can remove everything from $\mathcal{X}$ except the singular term t_+/y and the terms in which v_α is a lowest weight vector, annihilated by t_-. So we reduce to

$$\mathcal{X} = \frac{t_+}{y} + \sum_{\alpha \in P_-} \epsilon_\alpha v_\alpha y^{-m_\alpha}, \tag{3.23}$$

where P_- labels the lowest weight vectors.

The Slodowy slice $\mathcal{S}_{t_+}$ transverse to a nilpotent orbit $\mathcal{O}_{t_+}$ is defined to be the subspace of $\mathfrak{g}$ consisting of elements of the form

$$t_+ + \sum_{\alpha \in P_-} \epsilon_\alpha v_\alpha, \tag{3.24}$$

with arbitrary coefficients ϵ_α. $\mathcal{S}_{t_+}$ meets $\mathcal{O}_{t_+}$ in a single point (the point with all $\epsilon_\alpha = 0$) and has nice or "transverse" intersections with all orbits that it meets.

Clearly, functions $\mathcal{X}(y)$ of the form given in (3.23) are in one-to-one correspondence with points in the Slodowy slice $\mathcal{S}_{t_+}$; the correspondence is made by setting $y=1$ in (3.23). However, the moduli space $\mathcal{M}$ of vacua is not simply the Slodowy slice. We must impose the condition that the characteristic polynomial of $\mathcal{X}$ coincides with that of $\mathcal{X}_\infty$. The characteristic polynomial of $\mathcal{X}$ is independent of y because of the complex Nahm equation, so we can just evaluate this condition at $y=1$. We learn that $\mathcal{X}(1)$ takes values in the intersection of $\mathcal{S}_{t_+}$ with $\tilde{\mathcal{O}}_{\mathcal{X}_\infty}$, the subspace of $\mathfrak{g}$ consisting of elements with the same characteristic polynomial as $\mathcal{X}_\infty$. If $\mathcal{X}_\infty$ is regular, then $\tilde{\mathcal{O}}_{\mathcal{X}_\infty}$ is the same as $\mathcal{O}_{\mathcal{X}_\infty}$, the orbit of $\mathcal{X}_\infty$. In general, it is the closure $\overline{\mathcal{O}}_x$ of the orbit $\mathcal{O}_x$ of a regular element x with the same characteristic polynomial as $\mathcal{X}_\infty$.

What we learn, then, is that as a complex manifold, the moduli space $\mathcal{M}_\rho(\vec{X})$ of solutions of Nahm's equations with a pole determined by ρ is the intersection $\mathcal{S}_{t_+} \cap \overline{\mathcal{O}}_x$. In particular, from this we can determine the dimension of this space. If as before s denotes the number of summands when $\mathfrak{g}$ is decomposed in representations of $\mathfrak{su}(2)$, then the dimension of $\mathcal{S}_{t_+}$ is precisely s, since each irreducible representation of $\mathfrak{su}(2)$ has a one-dimensional space of lowest weight vectors. Requiring that $\mathcal{X}(1)$ should have the same characteristic polynomial as $\mathcal{X}_\infty$ reduces the complex dimension by r. So the dimension of $\mathcal{M}_\rho(\vec{X})$ is $s-r$.

3.3.1 Some Examples

At one extreme, if $t_+ = 0$, the corresponding transversal slice $\mathcal{S}_{t_+}$ is all of $\mathfrak{g}_\mathbb{C}$. So $\mathcal{M}_{\rho=0}(\vec{X}_\infty)$ is simply (if $\mathcal{X}_\infty$ is regular) the orbit $\mathcal{O}(\mathcal{X}_\infty)$, as before.

At the other extreme, if t_+ is a regular nilpotent element, the Slodowy slice $\mathcal{S}_{t_+}$ has dimension $s=r$, the rank of G. Its intersection with a regular orbit (or the closure of one) is therefore of dimension zero, and should consist of a finite set of points. But since the Slodowy slice $\mathcal{S}_{t_+}$ meets the regular orbit $\mathcal{O}_{t_+}$ in precisely one point (the element $t_+ \in \mathfrak{g}$), it likewise meets every regular orbit in just one point.

One can verify this by hand for $SL(n,\mathbb{C})$. A transversal to the orbit of the regular nilpotent element t_+ given in (3.14) that is not actually the Slodowy slice, but arises from it by a different gauge fixing of the gauge invariance (3.22), consists of elements of the form

$$x = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 \\ a_n & a_{n-1} & a_{n-2} & \dots & 0 \end{pmatrix}, \tag{3.25}$$

with coefficients $a_n, a_{n-1}, \dots, a_2, 0$ in the bottom row. (We set the lower right entry to zero to ensure that this matrix is in $\mathfrak{sl}(n,\mathbb{C})$; in $\mathfrak{gl}(n,\mathbb{C})$, this element would be another coefficient a_1.) Every set of values of the Casimir operators $\mathrm{Tr}\, x^k$, $k=2,\dots,n$ arise precisely once in this family. So this transversal slice meets every regular orbit precise once.

So we learn that if ρ corresponds to a regular nilpotent orbit, then the moduli space $\mathcal{M}_\rho(\vec{X}_\infty)$ consists of only a single point. As we will see, this result is important in understanding duality of supersymmetric boundary conditions.

For any other ρ, s is larger so the relevant moduli space has a positive dimension. For example, if G is simply-laced and t_+ is a subregular nilpotent element, then $s-r=2$ and the moduli space is of dimension 2. For $\mathcal{X}_\infty = 0$, it equals $\mathbb{C}^2/\Gamma$, where Γ is the finite subgroup of $SU(2)$ related to G, and for other $\mathcal{X}_\infty$, it is a deformation of $\mathbb{C}^2/\Gamma$. This is explained in [13].

3.4 Nahm's Equations and Brane Constructions

Now we will extend the analysis of Nahm's equations to allow for discontinuities as well as poles. Instead of proceeding in an abstract way, as we have done so far, we consider a specific (and well known) string theory situation. This is useful in understanding the action of duality, though in the present paper we take only limited steps in that direction.

We consider (Fig. 2) a system of n parallel D3-branes, transversely intersecting a D5-brane. The D3-branes are parametrized by x^0, x^1, x^2, x^3, and support a four-dimensional $U(n)$ gauge theory with $\mathcal{N} = 4$ supersymmetry; the values of $x^4, \ldots, x^9$ are observed in this theory as scalar fields $\vec{X}$ and $\vec{Y}$.

The D5-brane is supported at $x^3 = x^7 = x^8 = x^9 = 0$. The D5-brane supports a $U(1)$ gauge field. From the standpoint of the D3-brane system, which we will focus on, this $U(1)$ can be regarded as a global symmetry of the D3-brane theory (modulo a caveat noted below), and the fluctuations in the D5-brane position can be ignored.

In the D3-brane theory, there is a hypermultiplet Z in the fundamental representation of $U(n)$, supported at the intersection with the D5-brane. This intersection is at $x^3 = 0$; as usual, we write y for x^3. Z is a "bifundamental" hypermultiplet, meaning that it is also charged under the $U(1)$ symmetry coming from the D5-brane. But the action of this $U(1)$ is the same as that of the center of $U(n)$. Since $U(n)$ is gauged, this $U(1)$ is not really observed as a global symmetry of the D3-brane theory. Global symmetries will arise from D5-brane symmetries when there is more than one D5-brane, as in other cases that we treat below.

For the same reasons as in the examples that we have already treated, supersymmetric vacua of the combined system are given by solutions of Nahm's equations. However, we must include the contribution of Z in Nahm's equations. Essentially the same derivation[9] that led to (2.32), with $\vec{X}$ and $\vec{Y}$ exchanged, shows that the hyper-Kahler moment map for

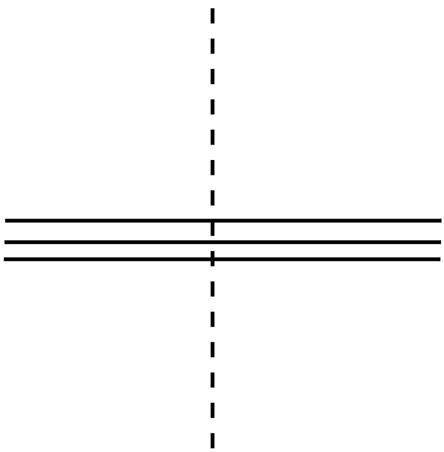

Fig. 2 Here and later, *horizontal solid lines* denote D3-branes whose world-volume is parametrized by x^0, x^1, x^2, x^3. *Vertical dotted lines* denote D5-branes supported at $x^3 = x^7 = x^8 = x^9 = 0$

[9] Because we are now on the full line $-\infty < y < \infty$, rather than the half-line $y \geq 0$, integration by parts does not produce a term $\delta(y)\vec{X}(0)$, which appears in the previous derivation.

the combined system consisting of hypermultiplets $\vec{X}$, $A=A_3$, and Z is

$$\vec{\mu}(y)=\frac{D\vec{X}}{Dy}+\vec{X}\times\vec{X}+\delta(y)\vec{\mu}^Z, \tag{3.26}$$

where $\vec{\mu}^Z$ is the hyper-Kahler moment map for the hypermultiplet Z. The extension of Nahm's equation is therefore

$$\frac{D\vec{X}}{Dy}+\vec{X}\times\vec{X}+\delta(y)\vec{\mu}^Z=0. \tag{3.27}$$

The meaning of the delta function is that $\vec{X}(y)$ is discontinuous at $y=0$. The jump $\Delta\vec{X}$ in crossing $y=0$ obeys

$$\Delta\vec{X}+\vec{\mu}^Z=0. \tag{3.28}$$

We now want solutions of this extended Nahm equation in which $\vec{X}$ approaches one limit $\vec{X}_{\infty,-}$ for $y\to-\infty$, and another limit, $\vec{X}_{\infty,+}$ for $y\to+\infty$. Of course, the components of $\vec{X}_{\infty,-}$ commute with each other, as do the components of $\vec{X}_{\infty,+}$. We want to describe the space $\mathcal{M}(\vec{X}_{\infty,-},\vec{X}_{\infty,+})$ of possible vacua of the combined system, for specified vacua at the far left and far right.

The usual arguments show that $\mathcal{M}$ is hyper-Kahler. But as in Sect. 3.2, a useful way to understand $\mathcal{M}$ is to describe it as a complex manifold in one of its complex structures. Proceeding in the usual way, we introduce the complex fields $\mathcal{X}=X_1+iX_2$, $\mathcal{A}=A+iX_3$, which obey a complex version of (3.27):

$$\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}y}+\delta(y)\mu_{\mathbb{C}}^Z=0. \tag{3.29}$$

Here $\mu_{\mathbb{C}}^Z=\mu_1^Z+i\mu_2^Z$ is the complex moment map of Z. $\mathcal{M}$ is the moduli space of solutions of this equation, with $\mathcal{X}(y)\to\mathcal{X}_{\infty,\pm}$ for $y\to\pm\infty$, and modulo complex gauge transformations that preserve this asymptotic condition. As before, the analysis is most straightforward if $\mathcal{X}_{\infty,\pm}$ are regular. (Actually, we will formulate the argument below in a way that remains valid as long as one of the two, say $\mathcal{X}_{\infty,-}$, is regular; a singularity develops only when both become non-regular.)

As usual, we can gauge away $\mathcal{A}$ by a complex gauge transformation $g(y)$ that does not necessarily preserve the asymptotic condition. In fact, we can set $g(-\infty)=1$, but then $g(\infty)$ may not commute with $\mathcal{X}_{\infty,+}$. (Of course, we can everywhere reverse the roles of $+\infty$ and $-\infty$.) A convenient way to proceed is to make a gauge transformation with $g(-\infty)=1$ that sets $\mathcal{A}=0$ everywhere. In this gauge, Nahm's equations reduce to

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}y}+\delta(y)\mu_{\mathbb{C}}^Z=0, \tag{3.30}$$

saying simply that $\mathcal{X}$ is piecewise constant, with a jump at $y=0$. Moreover, the boundary condition requires that $\mathcal{X}(y)=\mathcal{X}_{\infty,-}$ for $y<0$. After reducing Nahm's equation to this form with this boundary condition, we are still free to make a constant gauge transformation by an element of the group $T_{\mathbb{C}}$ that commutes with $\mathcal{X}_{\infty,-}$.

Now let us count parameters. A hypermultiplet in the fundamental representation of $U(n)$ has $2n$ complex parameters. After solving (3.30), we must impose n complex constraints to ensure that $\mathcal{X}(y)$ has the same characteristic polynomial as $\mathcal{X}_{\infty,+}$. We also remove n

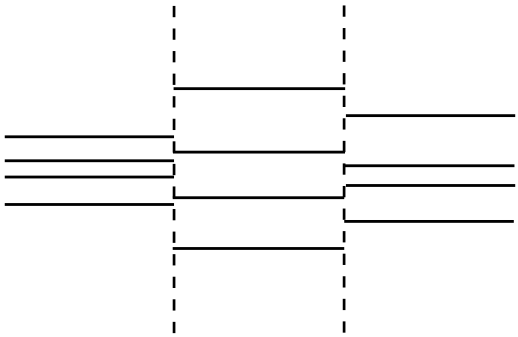

Fig. 3 A system of parallel D3-branes interacting with two parallel D5-branes. The D3-branes can "break" in crossing the D5-branes. The position of a D3-brane that connects two D5-branes is parametrized by the value of a hypermultiplet

parameters in dividing by the residual group $T_{\mathbb{C}}$ of gauge transformations. The net effect is that in this particular example, the moduli space $\mathcal{M}$ is zero-dimensional. In fact, it consists of precisely one point, as we will learn in Sect. 3.4.1.

Several D5-Branes We can apply similar methods to a more general problem with k D5-branes supported at points $y = y_\alpha$, $\alpha = 1, \ldots, k$. At each position y_α is supported a hypermultiplet Z_α in the fundamental representation of $U(n)$.

Nahm's equation now becomes

$$\frac{D\vec{X}}{Dy} + \vec{X} \times \vec{X} + \sum_{\alpha=1}^{k} \delta(y - y_\alpha)\vec{\mu}^{Z_\alpha} = 0. \tag{3.31}$$

After gauging $\mathcal{A}$ to zero and requiring that $\mathcal{X}(y) = \mathcal{X}_{\infty,-}$ for $y \ll 0$, the complex Nahm equation becomes

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}y} + \sum_{\alpha=1}^{k} \delta(y - y_\alpha)\mu_{\mathbb{C}}^{Z_\alpha} = 0. \tag{3.32}$$

So again, in this gauge $\mathcal{X}$ is piecewise constant, with jumps at $y = y_\alpha$, $\alpha = 1, \ldots, k$.

We count parameters as before. Each fundamental hypermultiplet Z_α contributes $2n$ parameters. We remove n parameters in dividing by the residual gauge symmetry, and n more for requiring that $\mathcal{X}(y)$ for $y \gg 0$ has the same characteristic polynomial as $\mathcal{X}_{\infty,+}$. So the total number of parameters is $2n(k-1)$.

So far it does not matter if the points y_α are distinct. If they are, there is actually a standard way to obtain the counting of parameters from a brane picture (Fig. 3). Between each pair of successive D5-branes, the n D3-branes can break away and move freely. The position of a D3-brane is part of a hypermultiplet, so this gives n hypermultiplets for each pair of successive D5-branes. With altogether k D5-branes, there are $k-1$ successive pairs, and so $n(k-1)$ hypermultiplets in all. A single hypermultiplet corresponds to 2 complex parameters, so there are $2n(k-1)$ complex parameters to specify the vacuum.

3.4.1 Uniqueness of the Vacuum

Going back to the case of a single D5-brane, we want to show that for prescribed $\vec{X}_{\infty,\pm}$, the vacuum is unique. Since we know that the moduli space $\mathcal{M}$ of vacua is of dimension zero, it consists of a finite set of points; it suffices to count these points in a special case. We will take $\mathcal{X}_{\infty,-} = \mathrm{diag}(S_1, S_2, \ldots, S_n)$, with all S_i distinct and nonzero, while $\mathcal{X}_{\infty,+} = 0$.

To analyze this situation, it helps to describe more explicitly the moment map of a fundamental hypermultiplet Z. From a complex point of view, Z consists of n chiral superfields B^i, $i=1,\dots,n$ in the fundamental representation of $U(n)$, and n such superfields C_j, $j=1,\dots,n$ in the antifundamental representation. The complex moment map is the rank 1 matrix[10] M whose matrix elements are $M^i{}_j = B^i C_j$. Putting the complex Nahm equation in the form (3.30), and writing $\mathcal{X}'$ and $\mathcal{X}''$ for the values of $\mathcal{X}$ for $y<0$ and $y>0$, respectively, we have

$$\mathcal{X}'' = \mathcal{X}' - M. \tag{3.33}$$

Of course, $\mathcal{X}' = \mathcal{X}_{\infty,-}$.

Since we are taking $\mathcal{X}_+ = 0$, we need $\mathcal{X}''$ to be nilpotent. The group $T_{\mathbb{C}} = (\mathbb{C}^*)^n$ of diagonal matrices can be used to set all components B^i (in the basis in which $\mathcal{X}_{\infty,-}$ is diagonal) to 1 or 0. If any of these matrix elements vanishes, it is impossible for $\mathcal{X}' - M$ to be nilpotent. For example, for $n=2$, if $B^1=0$, $B^2=1$, then $\mathcal{X}' - M$ takes the form

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ C_1 & C_2 \end{pmatrix}. \tag{3.34}$$

This matrix has S_1 as one of its eigenvalues and is not nilpotent.

So we take

$$B = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{3.35}$$

The condition that $\mathcal{X}' - M$ is nilpotent is equivalent to $\det(z - (\mathcal{X}' - M)) = z^n$. The left hand side in general equals $z^n + f_{n-1}z^{n-1} + \cdots + f_1 z + f_0$, and we must set the coefficients $f_{n-1},\dots,f_0$ to zero. These coefficients are linear functions of $C_1,\dots,C_n$ because M has rank 1. So there is precisely one solution.

3.4.2 One Extra Brane

Our next goal is to describe what happens when there are unequal numbers of D3-branes on the two sides of a D5-brane. We begin with the case that the difference is 1, say n D3-branes for $y<0$ and $n+1$ for $y>0$ (Fig. 4). First we describe what we claim is the appropriate description of this situation; then we will try to justify it.

What is depicted in Fig. 4 is an example of a supersymmetric domain wall interpolating between $\mathcal{N}=4$ super Yang-Mills theories with two different gauge groups—in the present case, $U(n)$ for $y<0$ and $U(n+1)$ for $y>0$. Such domain walls were discussed in Sect. 2.6. It turns out that from a field theory point of view, the supersymmetric domain wall of Fig. 4 can be described by the construction of Sect. 2.6.2 if we take $G = U(n+1)$ and $G' = U(n)$, and take H to be a copy of $U(n)$ that is a diagonal product of G' and a $U(n)$ subgroup of G. We also set $\tilde{G} = U(n) \times U(n+1)$.

For finding supersymmetric vacua, the relevant facts are as follows. There are no extra matter fields at $y=0$; a supersymmetric vacuum is to be described by solving Nahm's

[10] If $M: V \to V$ is a linear map, we define the rank of M to be the dimension of the image of M, that is, of the subspace MV of V.

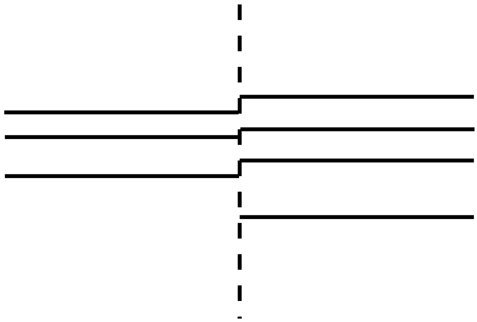

Fig. 4 In this example, the number of D3-branes jumps by 1 in crossing a D5-brane

equations for $\vec{X}$. For $y < 0$, the gauge group is $U(n)$ so the components of $\vec{X}$ are $n \times n$ matrices. But for $y \geq 0$, they are $(n + 1) \times (n + 1)$ matrices. What happens at $y = 0$ is simply that the smaller matrix is embedded as an $n \times n$ submatrix of the large one; the extra row and column are arbitrary.

For example, if $n = 2$, then $\vec{X}$ is a 2×2 matrix for $y < 0$:

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}. \tag{3.36}$$

For $y \geq 0$, an extra row and column appear:

$$\begin{pmatrix} * & * & \times \\ * & * & \times \\ \times & \times & \times \end{pmatrix}. \tag{3.37}$$

The upper left block is continuous at $y = 0$, and the other matrix elements are unconstrained at $y = 0$. Nahm's equations determine the dependence on y.

With some care, the recipe stated in the last paragraph can be extracted from (2.22). The most relevant part of (2.22) is $\vec{X}^+| = D_3\vec{X}^-| = 0$. As explained in Sect. 2.2.1, (2.22) can be used in $\mathcal{N} = 4$ super Yang-Mills theory with any gauge group $\tilde{G}$ and subgroup H, if one understands Φ^+ as the projection of an adjoint-valued field Φ from $\tilde{\mathfrak{g}}$ to $\mathfrak{h}$, and Φ^- as the projection to $\mathfrak{h}^\perp$. In the present case, we take $\tilde{G} = U(n) \times U(n + 1)$ and H the diagonal $U(n)$ subgroup described above. Then as described in Sect. 2.6, we "unfold" the theory to convert this boundary condition to a supersymmetric domain wall interpolating between gauge groups $U(n)$ and $U(n + 1)$. After unfolding, we arrive at the picture in the last paragraph.

An important detail is that in unfolding, we reverse the sign of $\vec{X}$ in one group, say $U(n + 1)$. So the condition that $\vec{X}^+| = 0$ in the folded theory is equivalent after unfolding to the statement that the $U(n)$ part of $\vec{X}$ is continuous at $y = 0$. This is the main claim in (3.37).

Now we wish to analyze the supersymmetric vacua in this situation. As in Sect. 3.4, we pick commuting triples $\vec{X}_{\infty,-}$ and $\vec{X}_{\infty,+}$ to specify choices of vacuum for large negative and large positive y. We denote as $\mathcal{M}(\vec{X}_{\infty,+}, \vec{X}_{\infty,-})$ the moduli space of vacua in the full system when the vacua at infinity are fixed. It is the moduli space of solutions of Nahm's equations (for matrices whose size jumps as above at $y = 0$) with $\vec{X}(y) \to \vec{X}_{\infty,\pm}$ for $y \to \pm\infty$. The allowed gauge transformations are by a function $g(y)$ that is $U(n)$-valued for $y < 0$ and $U(n + 1)$-valued for $y > 0$. At $y = 0$, $g(y)$ takes values in the subgroup $U(n)$ of $U(n + 1)$.

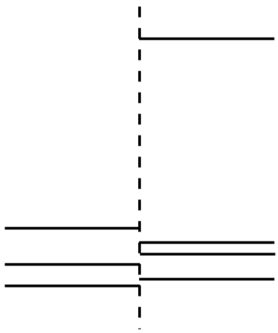

Fig. 5 By displacing a D3-brane that is on the right of the D5-brane very far from the others in the $x^4-x^5-x^6$ directions, we can reduce to a case with equals numbers of D3-branes on both sides

To describe $\mathcal{M}$, we use again its relation to the complex Nahm equation

$$0=\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}y}=\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}y}+[\mathcal{A},\mathcal{X}], \tag{3.38}$$

where $\mathcal{X}$ and $\mathcal{A}$ are complex-valued matrices whose size jumps at $y=0$ as above. As usual, in one of its complex structures, $\mathcal{M}$ is the moduli space of solutions of this equation such that $\mathcal{X}(y)\to\mathcal{X}_{\infty,\pm}$ for $y\to\pm\infty$, modulo complex gauge transformations. We analyze this problem in the familiar way by making a gauge transformation with $g(y)\to 1$ for $y\to-\infty$ to set $\mathcal{A}=0$. The rest of the argument is quite similar to steps we have already seen. Eqnuation (3.38) implies that $\mathcal{X}$ is constant for $y<0$ and hence equals $\mathcal{X}_{\infty,-}$. At $y=0$, it acquires $2n+1$ new complex coefficients (from the extra row and column). Of these, n can be removed by a gauge transformation that commutes with $\mathcal{X}_{\infty,-}$, and $n+1$ are fixed by requiring that $\mathcal{X}(y)$ for $y>0$ has the same characteristic polynomial as $\mathcal{X}_{\infty,+}$. So counting parameters, we see that the moduli space $\mathcal{M}$ is of dimension zero.

In fact, $\mathcal{M}$ consists of a single point. The argument for this closely follows Sect. 3.4.1. We assume that $\mathcal{X}_{\infty,-}$ is diagonal with distinct and nonzero eigenvalues and we take $\mathcal{X}_{\infty,+}=0$. By a constant gauge transformation that commutes with $\mathcal{X}_{\infty,-}$, we can set all matrix elements in the last column in (3.37) to 1 except the bottom one, and then the condition that $\mathcal{X}(y)$ is nilpotent for $y>0$ gives $n+1$ linear equations that uniquely determine the bottom row in (3.37).

Comparison to the D3-D5 System We now want to show that what has just been described is consistent with what we know about the D3-D5 system. (See [17] for a more thorough treatment of similar issues in the context of monopoles.)

If one of the D3-branes that are at $y>0$ in Fig. 4 moves far away, we reduce (Fig. 5) to the case of n D3-branes meeting a D5-brane. A bifundamental hypermultiplet must appear. Let us see how this happens.

We suppose that $\mathcal{X}_{\infty,+}$ has one large eigenvalue, which we call W, and we take $W\to\infty$ keeping all other eigenvalues of $\mathcal{X}_{\infty,+}$ fixed. (We also keep $\mathcal{X}_{\infty,-}$ fixed.) For instance, for $n=2$ we have

$$\mathcal{X}_{\infty,+}=\begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & W\end{pmatrix}, \tag{3.39}$$

and the matrix elements denoted $*$ will be held fixed while $W\to\infty$. We write $\mathcal{X}'_{\infty,+}$ for the upper left $n\times n$ block of $\mathcal{X}_{\infty,+}$.

After gauging $\mathcal{A}$ to zero and setting $\mathcal{X}(y) = \mathcal{X}_{\infty,-}$ for $y < 0$, we are supposed to pick the last row and column in (3.37) so that $\mathcal{X}(y)$, for $y > 0$, is conjugate to $\mathcal{X}_{\infty,+}$. In our example of $n = 2$, $\mathcal{X}_{\infty,-}$ is a 2×2 matrix that "grows" an extra row and column for $y > 0$. In fact, we pick the last row and column so that for $y > 0$

$$\mathcal{X}(y) = \begin{pmatrix} * & * & W^{1/2}B^1 \\ * & * & W^{1/2}B^2 \\ W^{1/2}C_1 & W^{1/2}C_2 & W \end{pmatrix}, \tag{3.40}$$

where the upper left block equals $\mathcal{X}_{\infty,-}$, and the coefficients B^i, C_j, $i, j = 1, \dots, n$ are kept fixed for $W \to \infty$.

Second order perturbation theory shows that for large W, one eigenvalue of $\mathcal{X}(y)$ equals W and the others are the eigenvalues of the $n \times n$ matrix

$$\mathcal{X}_{\infty,-} - M, \tag{3.41}$$

where M has matrix elements $M^i{}_j = B^i C_j$. Our problem is now to choose M so that this matrix is conjugate to $\mathcal{X}_{\infty,+}$. But this is precisely the problem that we encountered for the D3-D5 system, with the pair B^i, C_j playing the role of the bifundamental hypermultiplet. This shows how the physics of the D3-D5 intersection follows from our proposal concerning the asymmetric configuration with an extra D3-brane at $y > 0$.

Flowing in the Opposite Direction It is also possible to run this in reverse. We begin with a D3-D5 system with $n + 1$ D3-branes on each side of the D5-brane. Then we move one of the D3-branes at $y < 0$ very far from the others. We do this by giving $\mathcal{X}_{\infty,-}$ one large eigenvalue W, while keeping fixed its other eigenvalues as well as $\mathcal{X}_{\infty,+}$. We take $\mathcal{X}_{\infty,-} = \mathrm{diag}(w_1, w_2, \dots, w_n, W)$, where $w_1, \dots, w_n$ are the small eigenvalues. For large W, we should reduce to the problem with n D3-branes at $y < 0$ and $n + 1$ at $y > 0$.

As in (3.33), we are supposed to satisfy

$$\mathcal{X}_{\infty,-} - M = \mathcal{X}'', \tag{3.42}$$

where $\mathcal{X}''$ should be conjugate to $\mathcal{X}_{\infty,+}$ and in particular has all eigenvalues fixed as $M \to \infty$. For this, we take M of the form

$$M = \begin{pmatrix} \cdot & \cdot & \times \\ \cdot & \cdot & \times \\ \times & \times & W + \times \end{pmatrix} \tag{3.43}$$

(illustrated here for $n = 2$), where coefficients denoted $\times$ are kept fixed as $W \to \infty$, while coefficients denoted $\cdot$ vanish for $W \to \infty$ and are adjusted so that M is of rank 1. With this ansatz, the problem of satisfying (3.42) for large W is equivalent to what we got from Nahm's equations with matrices that jump in rank in crossing $y = 0$. The quantities labeled $\times$ in (3.43) simply map to the quantities labeled the same way in (3.37).

3.4.3 D3-Branes Ending on a D5-Brane

When the difference between the numbers of D3-branes on the two sides of a D5-brane exceeds 1, poles appear in the solutions of Nahm's equations. To isolate the essential subtleties, we begin with the extreme case of n D3-branes at $y > 0$ and none at $y < 0$ (Fig. 6a).

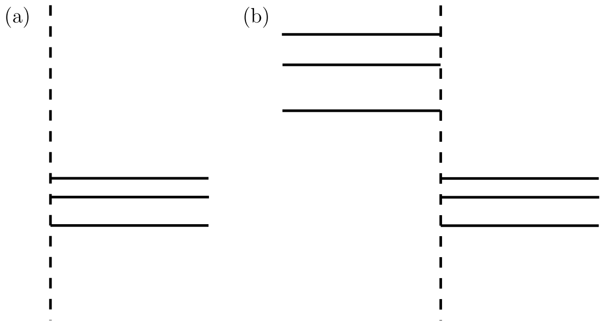

Fig. 6 (**a**) All D3-branes to the right of a D5-brane. (**b**) A configuration with equal numbers of D3-branes on both sides can be reduced to the one-sided configuration in (a) by moving all D3-branes on one side to large values of $x^{7,8,9}$

We approach this starting from the case of n D3-branes on each side, where everything is computable in weakly coupled string theory, and then we reduce to the case we want by removing the D3-branes on one side. To do this (Fig. 6b), we take the eigenvalues of $\vec{X}_{\infty,-}$ to be large, while $\vec{X}_{\infty,+}$ remains small or zero.

In the description by the complex Nahm equations, we use the usual gauge in which $\mathcal{X}$ is piecewise constant, equaling $\mathcal{X}'$ or $\mathcal{X}''$ for $y<0$ or $y>0$. $\mathcal{X}'$ must coincide with $\mathcal{X}_{\infty,-}$ (which we assume to be regular), and $\mathcal{X}''$ must have the same characteristic polynomial as $\mathcal{X}_{\infty,+}$. To achieve the situation depicted in Fig. 6b, we take the eigenvalues of $\mathcal{X}_{\infty,-}$ to be distinct and large, and we take $\mathcal{X}_{\infty,+}=0$. It follows that $\mathcal{X}''$ is nilpotent, and hence its rank is at most $n-1$.

Writing (3.33) in the form $\mathcal{X}'=\mathcal{X}''+M$, it says that the rank n matrix $\mathcal{X}'$ must be the sum of a rank 1 matrix M and the matrix $\mathcal{X}''$. Hence $\mathcal{X}''$ must have rank at least $n-1$. In view of the observation in the last paragraph, this means that $\mathcal{X}''$ has rank exactly $n-1$. Consequently, it is a regular nilpotent element, conjugate to the matrix in (3.14).

Let us suppose that $\vec{X}_{\infty,+}$ vanishes (and not just its complex part $\mathcal{X}_{\infty,+}$), so that $\vec{X}(y)\to 0$ for $y\to+\infty$. We know the form of a solution of Nahm's equations that approaches zero at infinity and for which $\mathcal{X}$ is a regular nilpotent. It is

$$\vec{X}=\frac{\vec{t}}{y+c^{-1}}, \tag{3.44}$$

where $\vec{t}$ are the generators of an irreducible $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(n)$, and c is a positive constant. Any solution on the half-line $y\ge 0$ with the stated properties has this form.

The constant c depends on the choice of $\mathcal{X}_{\infty,-}$. To reduce to the problem of D3-branes ending on a D5-brane (Fig. 6a), we wish $\mathcal{X}_{\infty,-}$ to have very large eigenvalues. Nahm's equations are invariant under the scaling $\vec{X}\to s\vec{X}$, $y\to s^{-1}y$, for positive s. Under this operation, we have $c\to sc$. So when we take $s\to\infty$ to send $\mathcal{X}_{\infty,-}$ to infinity, we also get $c\to\infty$. The limiting form of the solution for $y>0$ is then

$$\vec{X}=\frac{\vec{t}}{y}. \tag{3.45}$$

We have learned that the appropriate boundary condition for n D3-branes ending on a D5-brane at $y=0$ is that $\vec{X}$ must have a pole at $y=0$ corresponding to an irreducible $\mathfrak{su}(2)$ embedding. The same reasoning applies for a Dp-D$(p+2)$ system for any p. While this is a striking and perhaps surprising result, it has been discovered and explained in the past in several different ways. The original analysis of the Dp-D$(p+2)$ system and its relation to Nahm's equations and monopoles [3] implied this behavior, in view of the role of such poles in the theory of monopoles [1]. The pole has an elegant interpretation in terms of a distortion of the D5-brane by the "pull" of the D3-branes [18]; the different viewpoints have been related in [12].

What happens if we take $\vec{X}_{\infty,+}$ to be nonzero, and we keep it fixed while scaling $\vec{X}_{\infty,-}$ to infinity? As we learned in our study of the Slodowy slice, for every choice of $\vec{X}_{\infty,+}$, there is a unique solution of Nahm's equations on the half-line $y > 0$ that has the singularity of (3.45) for $y \to 0$ and approaches $\vec{X}_{\infty,+}$ for $y \to \infty$. This is the behavior that we will get for $y > 0$ in the situation just described.

3.4.4 A More General Case

Now we consider the general case of a D5-brane with m D3-branes ending on one side (Fig. 7) and $n > m$ on the other side. We have already treated the cases that $n = m + 1$, or $m = 0$. Here we assume that $n \geq m + 2 > 0$.

It is possible to guess what happens on the following grounds. We could simply remove m of the D3-branes by detaching them from the D5-brane and displacing them in $\vec{Y}$. This leaves $n - m$ D3-branes on one side of the D5-brane and none on the other side. In that case, we have just seen that a supersymmetric configuration of the remaining $n - m$ D3-branes is described by a solution of Nahm's equations with a pole associated to an irreducible embedding $\rho : \mathfrak{su}(2) \to \mathfrak{su}(n - m)$. Now if we move the extra D3-branes back, the simplest possibility is that they do not disturb this pole.

This suggests that the system is described by a solution of Nahm's equations with the following properties. For $y < 0$, $\vec{X}$ is an $m \times m$ matrix-valued solution of the equations, regular at $y = 0$. For $y > 0$, $\vec{X}$ is an $n \times n$ matrix-valued solution. Near $y = 0$, $\vec{X}$ looks like

$$\begin{pmatrix} \vec{A} & \vec{B} \\ \vec{C} & \vec{D} \end{pmatrix} \tag{3.46}$$

where the entries are as follows. $\vec{A}$ is an $m \times m$ matrix (or rather a trio of such matrices) and coincides with the limit of $\vec{X}(y)$ as y approaches zero from below. The lower right hand

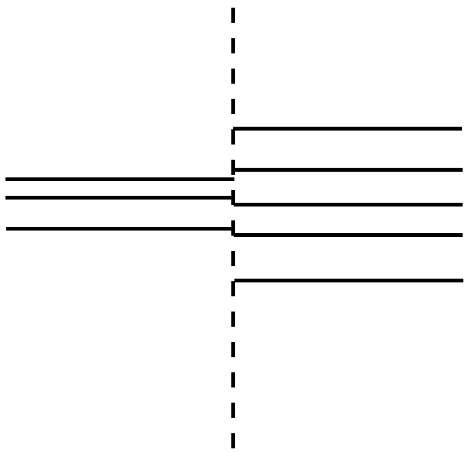

Fig. 7 Two extra D3-branes to the right of a D5-brane

block is an $(n-m)\times(n-m)$ block with

$$\vec{D}=\frac{\vec{t}}{y}+\cdots; \tag{3.47}$$

here $\vec{t}$ are generators of an irreducible embedding $\rho:\mathfrak{su}(2)\to\mathfrak{su}(n-m)$, and the ellipses are regular terms. The other blocks $\vec{B}$ and $\vec{C}$ are merely required to be regular for $y\to 0$.

This example is a domain wall of the type described in Sect. 2.6.2, with $G=U(n)$, $G'=U(m)$, a pole in Nahm's equations that breaks $G\times G'$ to $U(m)\times U(m)$, and H a diagonal subgroup of $U(m)\times U(m)$.

Notice that if we set $n-m=1$, then $\vec{t}=0$ (its components generate a trivial one-dimensional representation of $\mathfrak{su}(2)$) so there is no pole in (3.46), which actually then reduces to what we have analyzed in Sect. 3.4.2.

A Useful Trick Now let us discuss the solutions of Nahm's equations in this example. As in the other examples with only a single D5-brane, once one specifies $\vec{X}_{\infty,\pm}$, the relevant moduli space of solutions of Nahm's equation is zero-dimensional. In order to show this, we need to understand the effects of the pole at $y=0$. In a one-sided problem that we have already studied in Sect. 3.3, the analysis of the pole leads to the Slodowy slice transversal to a nilpotent orbit. Rather than making a similar analysis in a new situation, we will use a trick to reduce to the previous case. The trick in question also has other applications.

We let $\mathcal{M}_+$ denote the space of $n\times n$ solutions of Nahm's equations on the half-line $y>0$, with the form given in (3.46) near $y=0$ and approaching $\vec{X}_{\infty,+}$ (up to conjugacy) for $y\to\infty$. Thus, $\vec{X}$ has a pole at $y=0$ associated with an $\mathfrak{su}(2)$ embedding of rank $n-m$. The group $U(m)$ acts on $\mathcal{M}_+$, by gauge transformations at $y=0$ that commute with the pole.[11] The moment map for the action of $U(m)$ on $\mathcal{M}_+$ is

$$\vec{\mu}^{+}=\vec{A}, \tag{3.48}$$

where as in (3.46), $\vec{A}$ is the value at $y=0$ of the upper left block of $\vec{X}$. This formula was obtained in (3.4) (except that here we restrict to those global gauge transformations that commute with the pole). The complex dimension of $\mathcal{M}_+$ is $s-n$, where s is the number of summands when the Lie algebra $\mathfrak{u}(n)$ is decomposed in irreducible representation of $\mathfrak{su}(2)$ (embedded in $\mathfrak{u}(n)$ via $n=(n-m)+1+1+\cdots+1$). Performing this computation, we find that

$$\dim\mathcal{M}_+=m^2+m. \tag{3.49}$$

On the other hand, we can solve Nahm's equations in $m\times m$ matrices on the half-line $y\le 0$. Now we require the solution to be regular at $y=0$ and to approach $\vec{X}_{\infty,-}$ (up to conjugacy) for $y\to-\infty$. We denote the moduli space of solutions as $\mathcal{M}_-$. As a complex manifold, it is isomorphic to the orbit of $\mathcal{X}_{\infty,-}$ if that element is regular, and in any event its complex dimension is

$$\dim\mathcal{M}_-=m^2-m. \tag{3.50}$$

[11] The symmetry group is really $U(m)\times U(1)$, but the second factor will not be important. The moment map for the second factor is $\operatorname{Tr}\vec{D}$.

The group $U(m)$ acts on $\mathcal{M}_-$ by gauge transformations at $y=0$, and the hyper-Kahler moment map is

$$\vec{\mu}^- = -\vec{X}(0) = -\lim_{y\to 0^-}\vec{X}(y). \tag{3.51}$$

The reason for the minus sign is that we are now solving Nahm's equations on the half line $y \le 0$ rather than $y \ge 0$, and this reverses the sign of the endpoint contribution that results from integration by parts.

The product $\mathcal{M}_+ \times \mathcal{M}_-$ is a hyper-Kahler manifold acted on by $U(m)$. Let us take its hyper-Kahler quotient by $U(m)$. This entails setting to zero the combined moment map $\vec{\mu} = \vec{\mu}^+ + \vec{\mu}^-$ and dividing by $U(m)$. Setting $\vec{\mu}=0$ means that $\lim_{y\to 0^-}\vec{X}(0) = \vec{A}$. This means that the two partial solutions on the half-lines $y \le 0$ and $y \ge 0$ fit together to a solution on the whole line, with the right singularity at $y=0$ and the right matching condition as described in (3.46) to represent a supersymmetric vacuum of the full system. Also, in constructing $\mathcal{M}_+$ we have divided by gauge transformations for $y>0$, and in constructing $\mathcal{M}_-$ we have divided by gauge transformations for $y<0$. So after also dividing by $U(m)$ to construct the hyper-Kahler quotient of $\mathcal{M}_+ \times \mathcal{M}_-$ by $U(m)$, we have divided by all gauge transformations on the line.

The upshot is that the desired moduli space $\mathcal{M}$ of supersymmetric vacua of the combined system is the hyper-Kahler quotient of $\mathcal{M}_+ \times \mathcal{M}_-$ by $U(m)$, often denoted $(\mathcal{M}_+ \times \mathcal{M}_-)///U(m)$. Taking the hyper-Kahler quotient by a w-dimensional group reduces the complex dimension by $2w$. Since the dimension of $U(m)$ is m^2, we see, using (3.49) and (3.50), that $\mathcal{M}$ is zero-dimensional.

3.5 Pole of General Type

We are now ready to consider a much more general problem. We consider supersymmetric boundary conditions of D5-type in $U(n)$ gauge theory. From a field theory point of view, such a boundary condition can be constructed for any choice of a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{u}(n)$. In Sect. 3.4.3, we explained that the case that ρ is an irreducible $\mathfrak{su}(2)$ embedding corresponds to D3-branes ending on a D5-brane. This makes it relatively straightforward to understand the S-dual of this boundary condition.

As a basis for understanding S-duality in general, we would like to find a D-brane construction of the boundary condition associated to an arbitrary ρ. At first sight, this may appear difficult. A general ρ is specified by a decomposition $n = n_1 + n_2 + \cdots + n_k$, where the n_i are positive integers and we can assume that $n_1 \ge n_2 \ge \cdots \ge n_k$. How can we encode this information in terms of D-branes?

Roughly speaking, we do this by letting the n D3-branes end on k different D5-branes—with n_i D3-branes ending on the ith D5-brane, for $i = 1, \dots, k$. However, it is not clear what this is suppose to mean if all the D5-branes are located at $y=0$. In that case, how do we make sense of the question of which D3-brane ends on which D5-brane?

To make sense of it, we displace the D5-branes from each other, as in Fig. 8. Thus we consider a system with D5-branes at points y_α, $\alpha = 1, \dots, k$; we assume that $\tilde{n}_\alpha$ D3-branes end on the α^{th} D5-brane. Thus the total number of D3-branes is reduced by $\tilde{n}_\alpha$ when one crosses the α^{th} D5-brane from right to left. The numbers $\tilde{n}_\alpha$ will equal the n_i up to permutation—but it will be crucial, as we will see, to choose the right permutation.

From Sect. 3.4, we know what field theory construction corresponds to the configuration of Fig. 8. Supersymmetric vacua, for example, are described by a solution of Nahm's equations for matrices $\vec{X}(y)$ whose rank jumps whenever $y = y_\alpha$ for some α, and which have

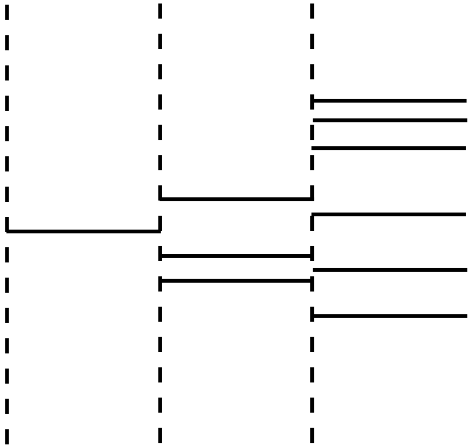

Fig. 8 To the *right* of this picture, there are six D3-branes. Reading from *right to left*, three of them end on the first D5-brane, two end on the second, and one ends on the third and last

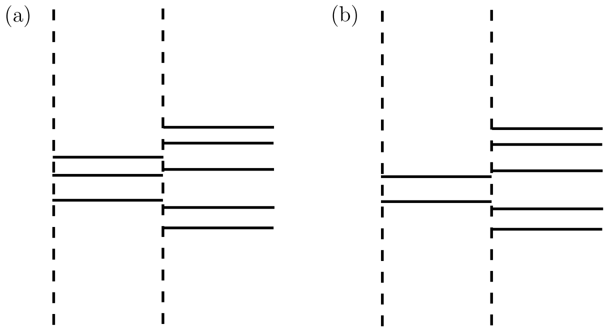

Fig. 9 To the *right* of the picture, there are five parallel D3-branes. Reading *from right to left*, in (**a**), two of them end on the first D5-brane and three on the second, while in (**b**), the numbers are reversed. This is the special case $n=5$, $n_1=3$, $n_2=2$ of a decomposition $n=n_1+n_2$

a pole for $y \to y_\alpha^+$ whenever $\tilde{n}_\alpha \geq 2$. Moreover, Fig. 8 is useful because it is described in terms of branes; this will be our starting point in a separate paper in analyzing its S-dual.

Our hypothesis then, is that for some choice of the $\tilde{n}_\alpha$, which will equal the n_i up to permutation, the brane configuration of Fig. 8 is equivalent, in the limit that all $y_\alpha \to 0$, to a field theory construction based on a corresponding $\mathfrak{su}(2)$ embedding. The $\mathfrak{su}(2)$ embedding, of course, is the one associated with the decomposition $n=n_1+n_2+\cdots+n_k$. In Sect. 3.5.1, we justify this claim by analyzing the moduli space of supersymmetric vacua. But first, we determine exactly how the $\tilde{n}_\alpha$ must be related to the n_i.

To explain the issue, we first consider the example $k=2$. Thus, there are precisely two D5-branes; n_1 end on one and n_2 on the other. There are two possible arrangements (Fig. 9), depending on whether the D5-brane on which the larger number of branes end is on the right or the left.

In Sect. 3.5.1, we will use Nahm's equations to describe the moduli space $\mathcal{M}$ of supersymmetric vacua (for a given limit $\vec{X}_\infty$ at infinity) in this situation. But for now, we count the parameters directly from the brane diagram.

Each D3-brane that is free to move between two D5-branes contributes one hyper-Kahler modulus or two complex moduli. In Fig. 9a, there are n_1 such branes and in Fig. 9b, there are n_2 such branes. (The figure is drawn for $n_1=3$, $n_2=2$.) So the number of complex moduli is $2n_1$ in one case and $2n_2$ in the other.

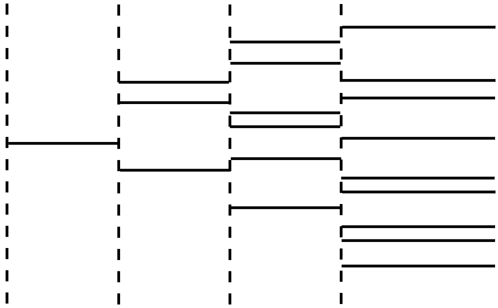

Fig. 10 Reading from right to left, the numbers of D3-branes ending on successive D5-branes are 3, 3, 2, and 1. This is a non-increasing sequence of numbers, so this configuration has a nice limit when the D5-branes become coincident

Let us compare this to a boundary condition in Yang-Mills theory on a half-space given by a solution of Nahm's equation with a pole at $y=0$. We suppose that the pole is determined by a homomorphism $\rho:\mathfrak{su}(2)\to\mathfrak{u}(n)$ associated with a decomposition $n=n_1+n_2$. The moduli space $\mathcal{M}$ of vacua, according to Sect. 3.3, has complex dimension $s-n$, where s is the number of summands when the Lie algebra $\mathfrak{u}(n)$ is decomposed as a direct sum of irreducible representations of $\mathfrak{su}(2)$. Assuming that $n_1\geq n_2$, one finds that $s=n+2n_2$. The complex dimension of $\mathcal{M}$ is therefore $2n_2$.

This shows that the configuration of Fig. 9b, but not that of Fig. 9a, may as $y_1,y_2\to 0$ approach the conformally invariant boundary condition determined by ρ. We believe this to be the case. We are not certain what is the limit for $y_1,y_2\to 0$ of the configuration of Fig. 9a, but we believe that it may be that the n_1-n_2 extra hypermultiplets simply decouple in this limit.

In Sect. 3.5.1, we will confirm the hypothesis about Fig. 9b by analyzing $\mathcal{M}$ as a complex manifold. For now, however, we just explain the counting for the case that ρ is an $\mathfrak{su}(2)$ embedding associated with a general decomposition $n=n_1+n_2+\cdots+n_k$, with $n_1\geq n_2\geq\cdots\geq n_k$. The number of irreducible $\mathfrak{su}(2)$ modules in the decomposition of $\mathfrak{u}(n)$ is $s=n+2\sum_{i<j}n_j=n+2\sum_{j=1}^k(j-1)n_j$. The hyper-Kahler dimension of $\mathcal{M}$ is therefore

$$(s-n)/2=\sum_{j=1}^{k}(j-1)n_j. \tag{3.52}$$

We stress that this is the dimension of $\mathcal{M}$ if and only if the n_i are labeled in non-ascending order.

Equation (3.52) agrees with the number of parameters suggested by the brane diagram if and only if, as one approaches the boundary from the bulk, the number of D3-branes ending on one D5-brane is at least as great as the number ending on the next one (Fig. 10). Indeed, the number of hyper-Kahler parameters suggested by the brane picture is $\sum_{j=1}^{k-1}b_j$, where b_j is the number of D3-branes between the jth and $j+1$th D5-brane. If $\tilde{n}_j$ D3-branes end on the jth D5-brane (counting them from right to left), then $b_j=\sum_{s=j+1}^k\tilde{n}_s$, so $\sum_{j=1}^{k-1}b_j=\sum_{j=1}^k(j-1)\tilde{n}_j$. Assuming that the numbers $\tilde{n}_j$ are supposed to be a permutation of the n_j, this number coincides with (3.52) if and only if $n_j=\tilde{n}_j$, so that the $\tilde{n}_j$ are non-ascending.

This has a heuristic explanation if one thinks of the D3-branes ending on a D5-brane as creating a "spike" in a D5-brane [18]. The condition of Fig. 9 then makes it possible for the various spikes to avoid intersecting each other (Fig. 11).

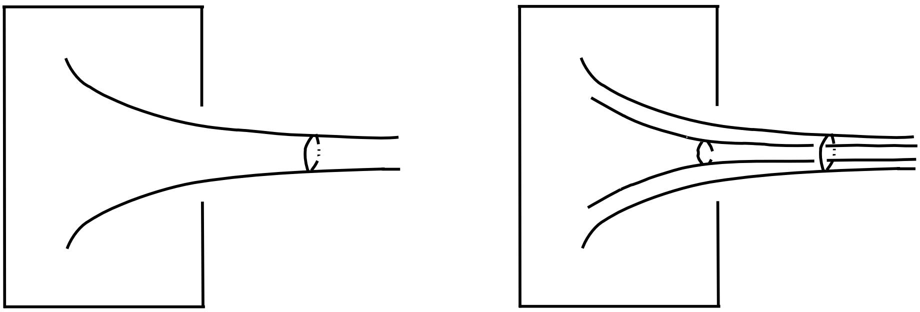

Fig. 11 (**a**) D3-branes ending on a D5-brane can be represented by a "spike." (**b**) The condition that the numbers of D3-branes ending on successive D5-branes are non-increasing (*from right to left*) ensures that one spike can fit inside the next

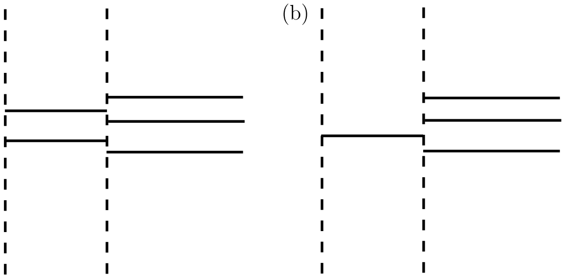

Fig. 12 These two configurations correspond to two different problems in $U(3)$ gauge theory, as described in the text. The configuration of (**b**) has a straightforward limit as the two NS5-branes approach each other, and that of (**a**) does not

3.5.1 Analysis of $\mathcal{M}$ as a Complex Manifold

We are now going to compare the complex manifolds associated with the brane diagrams of Figs. 10a and b to the answer coming from the corresponding $\mathfrak{su}(2)$ embedding. To make it easy to write explicit formulas, we will just describe the case $n = 3$, $n_1 = 2$, $n_2 = 1$ (Fig. 12). The general case is similar. We want to compare three complex manifolds:

(1) $\mathcal{M}$ parametrizes supersymmetric vacua in $U(3)$ gauge theory on the half-space $y \geq 0$, with a pole at $y = 0$ given by the $\mathfrak{su}(2)$ embedding (associated with the decomposition $3 = 2 + 1$), and with $\vec{X}(y) \to \vec{X}_\infty$ for $y \to \infty$.
(2) $\mathcal{M}'$ parametrizes supersymmetric vacua in the brane picture of Fig. 12a; one D3-brane ends on a D5-brane at $y = y_1 > 0$ and the other two continue to $y = 0$.
(3) $\mathcal{M}''$ parametrizes supersymmetric vacua corresponding to Fig. 12b; now two D3-branes end at $y = y_1$ and the other two continue to $y = 0$.

We can describe case (1) using (3.23). If we choose ρ to map $\mathfrak{su}(2)$ to matrices supported in the upper left 2×2 block of a 3×3 matrix (so that in particular the pole in $\mathcal{X}$ lives in that block), then the general form of a solution of the complex Nahm equations in case (1) is

$$\mathcal{X} = \begin{pmatrix} a & y^{-1} & 0 \\ by & a & cy^{1/2} \\ dy^{1/2} & 0 & e \end{pmatrix}, \tag{3.53}$$

where a, b, c, d and e are complex parameters. After fixing three parameters to specify the characteristic polynomial of $\mathcal{X}$, we see that $\mathcal{M}$ is two-dimensional.

In case (2), on the interval $0 \leq y \leq y_1$, we solve Nahm's equations via 2×2 matrices with a pole at $y = 0$. The general allowed form is again given by (3.23):

$$\mathcal{X} = \begin{pmatrix} a & y^{-1} \\ by & a \end{pmatrix}. \tag{3.54}$$

So far so good: this agrees with the upper left block in (3.53). However, when we cross $y = y_1$, the matrix simply grows a new row and column with no restriction on the new matrix elements:

$$\mathcal{X} = \begin{pmatrix} a & y^{-1} & c \\ by & a & d \\ e & f & g \end{pmatrix}. \tag{3.55}$$

Now there are seven complex parameters, and after adjusting three to fix the characteristic polynomial of $\mathcal{X}$, we find that $\mathcal{M}'$ has complex dimension four and hyper-Kahler dimension two. This agrees with what we would expect from the brane picture in Fig. 12a, and shows that $\mathcal{M}'$ cannot coincide with $\mathcal{M}$.

In case (3), on the interval $0 \leq y \leq y_1$, we solve Nahm's equations with 1×1 matrices. In particular, in the usual gauge, $\mathcal{X}$ is simply a complex constant e. Upon crossing $y = y_1$, two new rows and columns appear, and there is a pole at $y = y_1$ in the new 2×2 block. The general allowed form is

$$\mathcal{X} = \begin{pmatrix} a & (y - y_1)^{-1} & 0 \\ b(y - y_1) & a & c(y - y_1)^{1/2} \\ d(y - y_1)^{1/2} & 0 & e \end{pmatrix}. \tag{3.56}$$

The parameters correspond to those in (3.53) in an obvious way.

So the moduli space of vacua derived from the brane picture of Fig. 12b agrees with the one associated with the embedding $\rho : \mathfrak{su}(2) \to \mathfrak{u}(3)$. This makes it reasonable to expect that for $y_1 \to 0$, the model derived from the brane picture converges to $\mathcal{N} = 4$ super Yang-Mills theory with the superconformal boundary condition derived from ρ. This will be our starting point elsewhere in studying duality.

3.6 Moduli Space of Vacua with More General Boundary Conditions

Most of what we have done so far is to analyze Nahm's equations in the presence of a boundary condition associated to a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. The moduli space $\mathcal{M}_\rho$ (or $\mathcal{M}_\rho(\vec{X}_\infty)$) has turned out to be a Slodowy slice transverse to the raising operator t_+ associated to ρ, intersected with the closure of an orbit. A group F that is the commutant of ρ in G acts as a group of symmetries of $\mathcal{M}_\rho$. The hyper-Kahler moment map for the action of F is

$$\vec{\mu} = \vec{X}(0)_{\boldsymbol{f}}, \tag{3.57}$$

that is, the projection of $\vec{X}(0)$ to $\boldsymbol{f}$. The derivation of this statement is exactly the same as the derivation of (3.4), except that here we consider only gauge transformations that are F-valued at $y = 0$, so we project the formula to $\boldsymbol{f}$.

In Sect. 2.5, we described more general supersymmetric boundary conditions in which, roughly speaking, after picking ρ, we gauge a subgroup H of F and couple it to boundary degrees of freedom with H symmetry. Our next goal is to describe the moduli space of supersymmetric vacua in this more general context.

First we describe the effect of gauging H without adding boundary variables. Away from $y=0$, supersymmetry still requires that $\vec{X}$ should obey Nahm's equations. According to (2.45), the boundary condition requires $\vec{X}(0)^+=\vec{v}$, where $\vec{v}$ is a triple of elements of the center of $\mathfrak{h}$. (We omit the $\vec{\mu}^Z$ term as we are not yet including boundary variables.) We also must divide by the action of H, since this is now part of the gauge group. As the gauge symmetry has been reduced to H at the boundary, $\vec{X}(0)^+$ is just the projection of $\vec{X}(0)$ from $\boldsymbol{f}$ to $\mathfrak{h}$. So, according to (3.57), $\vec{X}^+(0)$ is a moment map for the action of H on $\mathcal{M}_\rho$. But as usual, we are free to add central elements to the moment map, and so $\vec{X}^+(0)-\vec{v}$ is an equally good moment map. The combined operation of setting $\vec{X}^+(0)=\vec{v}$ and dividing by H is therefore a hyper-Kahler quotient. Thus the moduli space $\mathcal{M}_{\rho,H}$ of solutions of Nahm's equations for the boundary condition associated to a general H is the same as the hyper-Kahler quotient by H of $\mathcal{M}_\rho$:

$$\mathcal{M}_{\rho,H}=\mathcal{M}_\rho///H. \tag{3.58}$$

The hyper-Kahler quotient is taken for a specified value of the FI constants $\vec{v}$.

There is no problem to add a boundary theory B with H action. B has its own moduli space of vacua, say $\mathcal{H}$, also a hyper-Kahler manifold with H action. If $\vec{\mu}_B$ is the moment map for the action of H on $\mathcal{H}$, then the incorporation of the boundary variables has the effect of replacing the boundary condition $\vec{X}(0)^+=0$ by $\vec{X}(0)^++\vec{\mu}_B=0$. This was demonstrated in (2.33).

So when boundary variables are added, we construct the moduli space of vacua by beginning with $\mathcal{M}_\rho\times\mathcal{H}$, imposing the boundary condition $\vec{X}(0)^++\vec{\mu}_B=\vec{v}$, and dividing by H. On the other hand, $\vec{X}(0)^++\vec{\mu}_B-\vec{v}$ is a moment map for the action of H on $\mathcal{M}_\rho\times\mathcal{H}$, so setting this to zero and dividing by H amounts to a hyper-Kahler quotient. The moduli space of vacua of the combined system is therefore

$$\mathcal{M}_{\rho,H,B}=(\mathcal{M}_\rho(\vec{X}_\infty)\times\mathcal{H})///H. \tag{3.59}$$

3.7 Including $\vec{Y}_\infty$

We have written explicitly the dependence of $\mathcal{M}_\rho$ on $\vec{X}_\infty$ in (3.59) to emphasize that this analysis does incorporate the value of $\vec{X}$ at infinity. However, throughout this section, we have taken $\vec{Y}_\infty=0$. It is now time to incorporate $\vec{Y}_\infty$. In doing so, we assume that $\vec{X}_\infty$ and $\vec{Y}_\infty$ taken together are regular and break G to a maximal torus T.

Supersymmetry requires that in bulk $\vec{Y}$ must obey (2.41)

$$\frac{D\vec{Y}}{Dy}=[\vec{Y},\vec{Y}]=[\vec{Y},\vec{X}]=0, \tag{3.60}$$

and thus, the components of $\vec{Y}$ generate symmetries of the solution of Nahm's equations. We must supplement this with additional information associated with the boundary conditions.

As summarized in Sect. 2.5, the most general half-BPS boundary conditions depends on the choice of a triple (ρ,H,B), where $\rho:\mathfrak{su}(2)\to\mathfrak{g}$ is a homomorphism, H is a subgroup of G that commutes with ρ, and B is a boundary theory with H symmetry. To explain the main points most directly, we first suppose that ρ and B are trivial.

The boundary condition on $\vec{Y}$ was described in (2.45). We decompose $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, where $\mathfrak{g}^+ = \mathfrak{h}$ and $\mathfrak{g}^-$ is the orthocomplement. Then we pick elements $\vec{w} \in \mathfrak{g}^-$ that commute with each other and with $\mathfrak{h}$, and require

$$\vec{Y}^-(0) = \vec{w}. \tag{3.61}$$

Equivalently, we require

$$\vec{Y}(0) = \vec{w} \bmod \mathfrak{h}. \tag{3.62}$$

This equation plus the covariant constancy of $\vec{Y}$, which is part of (3.60), says that $\vec{Y}_\infty$ must be conjugate to $\vec{w}$ mod $\mathfrak{h}$. If this is not the case, then the moduli space of supersymmetric vacua is empty. For most choices of H, that is the situation for a generic choice of $\vec{Y}_\infty$ and $\vec{w}$. For example, suppose that H is trivial and $\vec{w} = 0$. Then the condition is that $\vec{Y}_\infty$ must be conjugate to 0, that is, it must vanish. Otherwise, there are no supersymmetric vacua.

If $\vec{Y}_\infty$ is conjugate to $\vec{w}$, this may be so in inequivalent ways. For example, suppose that $G = SU(4)$ and $H = SU(2)$, consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}. \tag{3.63}$$

Let $\vec{w} = \mathrm{diag}(\vec{a}, -\vec{a}, 0, 0)$ (so $\vec{w} \in \mathfrak{g}^-$, and its components commute with each other and with H), $\vec{X} = \mathrm{diag}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$, $\vec{Y} = \mathrm{diag}(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4)$. Conjugating $\vec{Y}$ to equal $\vec{w}$ mod $\mathfrak{h}$ means finding i and j such that $\vec{y}_i = \vec{a}$ and $\vec{y}_j = -\vec{a}$. Generically this cannot be done (and the moduli space of supersymmetric vacua is empty), but it can happen that there is more than one way to do this. (This occurs if the $\vec{y}_i$ are pairwise equal, but $\vec{x}_i$ is generic enough that the collection $\vec{X}, \vec{Y}$ is regular.) When that is the case, each choice leads potentially to a component of the moduli space of vacua.

Making a particular choice of how to conjugate $\vec{Y}_\infty$ to be in the form

$$\vec{Y}_\infty = \vec{w} + h, \tag{3.64}$$

with $h \in H$, let us describe the associated component of the moduli space. It is convenient to think of H as a fixed subgroup of G, and $\vec{w}$ and $\vec{Y}_\infty$ as fixed elements of $\mathfrak{g}$, rather than all this being given up to conjugacy.

Now in solving Nahm's equations, we have worked in a gauge with $A_3 = 0$. Since $\vec{Y}$ is covariantly constant, it is actually constant in this gauge. According to (3.60), the solution of Nahm's equations takes values in the subgroup $G_{\vec{Y}}$ of G that commutes with $\vec{Y}$. This has the important consequence that for all y, $\vec{X}^+(y)$ takes values in $H_{\vec{Y}}$, the subgroup of H that commutes with $\vec{Y}$. This is so even before we specialize to the boundary, $y = 0$.

In Sect. 3.6, we found that the effect of having H non-trivial is that, after constructing the moduli space of solutions of Nahm's equations, we must take the hyper-Kahler quotient by the action of H. There are two related reasons that this is not the right thing to do when $\vec{Y}_\infty \neq 0$.

First, a key part of the hyper-Kahler quotient was to set to zero the hyper-Kahler moment map $\vec{X}^+(0) - \vec{v}$. (We recall that $\vec{v}$ are constants valued in the center of $\mathfrak{h}$.) In the present context, part of $\vec{X}^+(y)$ already vanishes before setting $y = 0$, namely the part that does not commute with $\vec{Y}$. It only makes sense at the boundary to add a condition on the projection of

$\vec{X}^+$ to $\mathfrak{h}_{\vec{Y}}$ (the centralizer of $\vec{Y}$ in $\mathfrak{h}$). So we may as well regard the constraint $\vec{X}^+(0)-\vec{v}=0$ as an equation in $\mathfrak{h}_{\vec{Y}}$, the Lie algebra of $H_{\vec{Y}}$. This is the moment map for the action of $H_{\vec{Y}}$, not the action of H.

Second, the other part of the hyper-Kahler quotient is to divide by H. But, with $\vec{Y}\neq 0$ and equal to a specified constant, there is no H symmetry, so we cannot divide by H. We can divide only by $H_{\vec{Y}}$.

The conclusion from each of the last two paragraphs is that what we want is a hyper-Kahler quotient by $H_{\vec{Y}}$. After making a choice α of how to put $\vec{Y}_\infty$ in the form (3.64), we should construct the corresponding moduli space $\mathcal{M}^\alpha_{\vec{Y}}(\vec{X}_\infty)$ of $G_{\vec{Y}}$-valued solutions of Nahm's equations in which $\vec{X}$ is conjugate at infinity (by an element of $G_{\vec{Y}}$) to $\vec{X}_\infty$, and take its hyper-Kahler quotient by $H_{\vec{Y}}$. After summing over α, we get the moduli space of vacua:

$$\mathcal{M}_{H,\vec{Y}}=\bigcup_\alpha \mathcal{M}^\alpha_{\vec{Y}}///H_{\vec{Y}}. \tag{3.65}$$

This discussion is not changed in an essential way by including a homomorphism $\rho:\mathfrak{su}(2)\to\mathfrak{g}$ or a boundary CFT with H symmetry. The boundary CFT just gives another factor $\mathcal{H}$ (its moduli space of vacua) that must be included in the hyper-Kahler quotient. The effect of ρ is just that, as usual, in solving Nahm's equations we must require $\vec{X}(y)$ to have a pole at $y=0$. Equation (3.64) in any case implies that $\vec{Y}$ commutes with ρ.

3.8 Duality: First Steps

Though we defer a serious study of how duality acts on half-BPS boundary conditions to a subsequent paper, we make here some preliminary observations that may help place in context some of the constructions that we have described.

A basic question is to ask what is the S-dual of the simplest Neumann boundary conditions, described in Sect. 2. With these boundary conditions, the vacuum is uniquely determined if one specifies the value of $\vec{X}$ at infinity. For fixed $\vec{X}_\infty$, the moduli space of supersymmetric vacua consists of only a single point.

The duality transformation $S:\tau\to -1/\tau$ transforms the unbroken supersymmetries of $\mathcal{N}=4$ super Yang-Mills in a non-trivial fashion, which is described for example in (2.25) of [19]. For simplicity, let us specialize to the case that τ is imaginary (or in other words the case that the θ angle vanishes). The transformation of the unbroken supersymmetries is then

$$\varepsilon\to\frac{1-\Gamma_{0123}}{\sqrt{2}}\varepsilon, \tag{3.66}$$

and this has the effect of exchanging the supersymmetry preserved with Neumann boundary conditions with the supersymmetry preserved by Dirichlet conditions. (The generalization of (3.66) to $\theta\neq 0$ is given in (4.35).)

One might think that the dual of Neumann boundary conditions would be Dirichlet boundary conditions. This, however, cannot be the case, because Dirichlet boundary conditions lead to a non-trivial moduli space of solutions of Nahm's equations, which has no analog for the Neumann case. The dual of Neumann boundary conditions must be a boundary condition which preserves the same supersymmetry as Dirichlet, but which does not lead to a non-trivial moduli space.

We have learned that for each choice of $\rho:\mathfrak{su}(2)\to\mathfrak{g}$, we can generalize Dirichlet boundary conditions to a more general boundary condition that preserves the same supersymmetry. The resulting moduli space of solutions of Nahm's equations is a Slodowy slice associated to ρ, and is trivial if and only if ρ is the principal $\mathfrak{su}(2)$ embedding. So this, rather than naive Dirichlet boundary conditions, is the natural candidate for the S-dual of Neumann boundary conditions.

For the case of $G = U(N)$, this proposal can be confirmed by considering the D3-NS5 and D3-D5 systems.[12] For N D3-branes ending on an NS5-brane, we get $U(N)$ gauge theory with Neumann boundary conditions. The S-dual consists of N D3-branes ending on a D5-brane. As we have learned, this corresponds not to naive Dirichlet boundary conditions but to Dirichlet boundary conditions modified with the principal embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. (This fact seems to underlie many occurrences of the principal $\mathfrak{su}(2)$ embedding in the geometric Langlands program.)

A converse question is to ask what is the S-dual of ordinary Dirichlet boundary conditions (with $\rho = 0$). The answer involves Neumann boundary conditions modified by coupling to a certain boundary superconformal field theory. To elucidate the nature of this theory, and to answer analogous questions for other boundary conditions described in Sect. 2, will be our goal in a separate paper.

3.9 Other Moduli Spaces of Solutions of Nahm's Equations

In our study of Nahm's equations so far, the goal has been to describe the moduli space of vacua of gauge theory on a half-space $y \geq 0$, with BPS boundary conditions at $y = 0$ and specified values of $\vec{X}$ and $\vec{Y}$ at $y = \infty$. Here we will briefly describe some other related spaces of solutions of Nahm's equations. (Apart from their intrinsic interest, these are relevant to a more detailed study of S-duality of boundary conditions that will appear elsewhere.)

3.9.1 Hyper-Kahler Analog of a Lie Group

The most basic of these [13] is the moduli space of solutions of Nahm's equations on a finite interval $0 \leq y \leq \ell$. We consider the gauge-invariant form of Nahm's equations

$$\frac{D\vec{X}}{Dy} + \vec{X} \times \vec{X} = 0 \tag{3.67}$$

for a pair $\vec{X}$, A, modulo gauge transformations that equal 1 at both $y = 0$ and $y = \ell$. By the usual reasoning, the moduli space, which we will call $\mathcal{G}_\ell$, is a hyper-Kahler manifold. (A simple scaling argument shows that the ℓ-dependence of the hyper-Kahler metric of $\mathcal{G}_\ell$ is a simple factor of $1/\ell$. The same is true of the generalizations introduced below.) Moreover, the group $G \times G$ acts on $\mathcal{G}_\ell$. One copy of G acts by gauge transformations at $y = 0$ and the second copy acts by gauge transformations at $y = \ell$. We write G_L and G_R for G acting on the left or right, that is at $y = 0$ or at $y = \ell$.

We can calculate the moment map $\vec{\mu}_L$ and $\vec{\mu}_R$ for the left and right action of G as in (3.4), leading to

$$\begin{aligned}\vec{\mu}_L &= \vec{X}(0),\\ \vec{\mu}_R &= -\vec{X}(\ell).\end{aligned} \tag{3.68}$$

[12] For the other classical groups $SO(N)$ and $Sp(N)$, a similar argument can be given by combining the branes with an orientifold threeplane.

(The minus sign in the second line comes in integration by parts.) As an example of the use of this formula, let us compute the hyper-Kahler quotient $\mathcal{G}///G_R$. We do this by setting $\vec{\mu}_R=0$ and dividing by G_R. Since $\vec{\mu}_R=\vec{X}(\ell)$ and Nahm's equations are of first order in $\vec{X}$, a solution with $\vec{\mu}_R=0$ has $\vec{X}$ identically zero. Dividing by G_R, after already dividing by gauge transformations that are 1 at $y=0,\ell$, means that we divide by all gauge transformations that are 1 at $y=0$. This enables us to set $A=0$ in a unique fashion. So the hyper-Kahler quotient of $\mathcal{G}$ by G_R—or likewise its hyper-Kahler quotient by G_L—is a single point.

As a second example, pick two positive numbers ℓ and ℓ', and consider the group G acting on the right on $\mathcal{G}_\ell$ and on the left on $\mathcal{G}_{\ell'}$. We claim that the hyper-Kahler quotient, which we abbreviate as $\mathcal{G}_\ell\times_G\mathcal{G}_{\ell'}$, is simply $\mathcal{G}_{\ell+\ell'}$. To get this result, we think of $\mathcal{G}_\ell$ as the moduli space of pairs $\vec{X},A$ that obey Nahm's equations on the interval $[0,\ell]$, and $\mathcal{G}_{\ell'}$ as the moduli space of pairs $\vec{X}',A'$ that obey Nahm's equations on the interval $[\ell,\ell+\ell']$. In each case we divide by gauge transformations that are 1 on the boundary of the interval. To compute the hyper-Kahler quotient by the diagonal product of the right action of G on the first factor and the left action on the second factor, we set to zero the moment map, which is $\hat{\mu}=-\vec{X}(\ell)+\vec{X}'(\ell)$, and then divide by gauge transformations acting on all fields at $y=L$. Once we set $\hat{\mu}=0$ and divide by gauge transformations at $y=L$, the quantities $\vec{X},A$ and $\vec{X}',A'$ fit together to a single solution of Nahm's equations on the full interval $[0,\ell+\ell']$, modulo gauge transformations that are 1 on the boundary. Hence

$$\mathcal{G}_\ell\times_G\mathcal{G}_{\ell'}=\mathcal{G}_{\ell+\ell'}. \tag{3.69}$$

In any one of its complex structures, $\mathcal{G}_\ell$ is isomorphic to $T^*G_\mathbb{C}$, the cotangent bundle of the complex Lie group $G_\mathbb{C}$ (in particular, as a complex manifold, it is independent of ℓ). To see this, as usual we introduce the variables $\mathcal{X}=X_1+iX_2$ and $\mathcal{A}=A+iX_3$. In one of its complex structures, $\mathcal{G}_\ell$ is equivalent to the moduli space of solutions of the complex Nahm equation

$$\frac{\mathcal{D}\mathcal{X}}{\mathcal{D}y}=0 \tag{3.70}$$

modulo complex-valued gauge transformations that equal 1 at $y=0,\ell$. The gauge-invariant data characterizing this solution is $\mathcal{X}(0)$ and the "Wilson line" or holonomy

$$g=P\exp\left(-\int_0^\ell\mathcal{A}\right). \tag{3.71}$$

(We need not include $\mathcal{X}(\ell)$ since the complex Nahm equation implies that it coincides with $g\mathcal{X}(0)g^{-1}$.) Here g takes values in $G_\mathbb{C}$ and we can consider $\mathcal{X}(0)$ to take values in the cotangent bundle of $G_\mathbb{C}$ at the point g. They are subject to no additional restrictions, so we can identify $\mathcal{G}_\ell$ holomorphically, in any one of its complex structures, with $T^*G_\mathbb{C}$.

The subgroup of $G\times G$ leaving fixed a given point in $\mathcal{G}$ is always a subgroup of G, with a diagonal embedding in $G\times G$. In fact, a symmetry of a solution of Nahm's equations must be generated by a covariantly constant gauge parameter, which is determined by its restriction to $y=0$. Any solution with the full G symmetry has $\vec{X}=0$. The gauge-invariant data contained in the solution is then the G-valued holonomy $P\exp(-\int_0^\ell A)$. This may be any element of G, so solutions with $\vec{X}=0$ furnish a copy of G embedded in $\mathcal{G}_\ell$, and these are the solutions for which the unbroken symmetry is maximal. By identifying a solution of Nahm's equations with the initial values $\vec{X}(0)$ plus the holonomy $P\exp(-\int_0^\ell A)$, one can show that as a manifold with $G\times G$ action, $\mathcal{G}$ is equivalent to $G\times\mathfrak{g}^3$, with a natural action of $G\times G$.

3.9.2 Hyper-Kahler Analog of a Homogeneous Space

The space $\mathcal{G}_\ell$ is in a sense the hyper-Kahler analog of a Lie group. There is also [15] a hyper-Kahler analog of a homogeneous space. For this, we solve Nahm's equations on the interval $[0,\ell)$, modulo gauge transformations that are 1 on the boundary, but now we pick a homomorphism $\rho:\mathfrak{su}(2)\to\mathfrak{g}$ and we require that $\vec{X}$ should have a pole of type ρ at $y=\ell$:

$$\vec{X}\sim\frac{\vec{t}}{y-\ell}. \tag{3.72}$$

(As usual, $\vec{t}$ is the image under ρ of a standard set of $\mathfrak{su}(2)$ generators.) We call the moduli space $\mathcal{T}_\ell^\rho$. It is a hyper-Kahler manifold, as usual. The group G acts by gauge transformations at $y=0$, with moment map

$$\vec{\mu}=\vec{X}(0). \tag{3.73}$$

The hyper-Kahler quotient $\mathcal{T}_\ell^\rho///G$ is empty, because (for non-zero ρ) it is impossible to solve Nahm's equations with the initial condition $\vec{X}(0)=0$ and with the polar behavior (3.72) at $y=\ell$.

We denote as $\mathcal{G}_\ell\times_G\mathcal{T}_{\ell'}^\rho$ the hyper-Kahler quotient of $\mathcal{G}_\ell\times\mathcal{T}_{\ell'}^\rho$ by G, with G acting on the right on the first factor and as just stated on the second factor. The same steps that led to (3.69) give

$$\mathcal{G}_\ell\times_G\mathcal{T}_{\ell'}^\rho=\mathcal{T}_{\ell+\ell'}^\rho. \tag{3.74}$$

If ρ has a nontrivial centralizer $H\subset G$, then $\mathcal{T}_\ell^\rho$ admits an action of H, by gauge transformations at $y=\ell$, commuting with the action of G. The moment map is $\vec{\mu}_H=\vec{X}(\ell)_{\mathfrak{h}}$, where $\vec{X}(\ell)_{\mathfrak{h}}$ is the projection of $\vec{X}(\ell)$ to $\mathfrak{h}$. The unbroken subgroup of $G\times H$ at any point in $\mathcal{T}_\ell^\rho$ is a subgroup of H, with a diagonal embedding in $H\times H\subset G\times H$. To see this, observe that a symmetry that leaves fixed a given solution of Nahm's equations is generated by a gauge parameter that is covariantly constant, and whose restriction to $y=\ell$ must commute with the Nahm pole. A solution whose unbroken symmetry is actually H can be obtained by setting $A=0$ and $\vec{X}=\vec{t}/(y-\ell)$.

Just like $\mathcal{G}_\ell$, $\mathcal{T}_\ell^\rho$ can be described explicitly as a complex manifold in any one of its complex structures. It is parametrized by a pair (g,η), where $g\in G_{\mathbb{C}}$ and $\eta\in\mathfrak{g}$ is a lowest weight vector with respect to ρ (in other words, $[\rho(t_-),\eta]=0$). We cannot quite define g as the holonomy operator (3.71); this holonomy does not converge, since $\mathcal{A}$ has a pole at $y=\ell$. Instead, we define g as a regularized version of the holonomy:

$$g=\lim_{\delta\to 0}\left[(-\delta)^{it_3}P\exp\left(-\int_0^{\ell-\delta}\mathcal{A}\right)\right]. \tag{3.75}$$

Similarly, η is defined as a regularized version of $\mathcal{X}(\ell)$. (It is simpler to use $\mathcal{X}(\ell)$ rather than $\mathcal{X}(0)$, since the conditions that it obeys are more simply stated.) In a gauge in which $\mathcal{A}=it_3/(y-\ell)$ in a neighborhood of $y=\ell$, the solution for $\mathcal{X}$ is given essentially by (3.23):

$$\mathcal{X}(y)=\frac{t_+}{y-\ell}+\sum_\alpha\epsilon_\alpha v_\alpha(y-\ell)^{-m_\alpha}, \tag{3.76}$$

where ϵ_α are complex constants and the v_α are a basis of lowest weight vectors with $[it_3,v_\alpha]=m_\alpha v_\alpha$, $m_\alpha\le 0$. So we define

$$\eta=(-\delta)^{it_3}(\mathcal{X}(\ell-\delta)+t_+/\delta)(-\delta)^{-it_3}, \tag{3.77}$$

which is independent of δ for small δ.

We can go one step farther and define $\mathcal{S}_\ell^{\rho,\rho'}$ to be the moduli space of solutions of Nahm's equations on the interval $[0,\ell]$ with poles of type ρ and ρ', respectively, at the two endpoints. (These are the boundary conditions used by Nahm in the original work [1] relating Nahm's equations to BPS monopoles.) This more general moduli space can be constructed from the ones that we have already considered as a hyper-Kahler quotient:

$$\mathcal{S}_{\ell+\ell'}^{\rho,\rho'} = (\mathcal{T}_\ell^\rho \times \mathcal{T}_{\ell'}^{\rho'})///G. \tag{3.78}$$

This can be shown by following the derivation of (3.69).

3.9.3 Including an NS5-Brane

Our last topic is to consider what happens to Nahm's equations in the presence of an NS5-brane.[13]

We suppose that the NS5-brane is located at $y=0$. We assume that there are n D3-branes ending on this NS5-brane on its left, and m on its right. The low energy physics is well known. For $y<0$, there is a $U(n)$ gauge theory with $\mathcal{N}=4$ supersymmetry. For $y>0$, the gauge group is $U(m)$. At $y=0$, there is a bifundamental hypermultiplet of $U(n)\times U(m)$. We write Z for the space parametrized by the bifundamental hypermultiplet and $\vec{\mu}_L^Z$, $\vec{\mu}_R^Z$ for the moment maps for the action on Z of $U(n)$ and $U(m)$, respectively.

Similarly, we write $\vec{X}_L$ and $\vec{X}_R$ for the fields $\vec{X}$ for $y<0$ and $y>0$, respectively. Like $\vec{\mu}_L^Z$ and $\vec{\mu}_R^Z$, they take values in the adjoint representations of $U(n)$ and $U(m)$, respectively. In a supersymmetric configuration, $\vec{X}_L$ and $\vec{X}_R$ must obey Nahm's equations away from $y=0$. The appropriate boundary conditions at $y=0$ are special cases of (2.33):

$$\begin{aligned} -\vec{X}_L(0)+\vec{\mu}_L &= 0, \\ \vec{X}_R(0)+\vec{\mu}_R &= 0. \end{aligned} \tag{3.79}$$

(The minus sign in the first line comes from integrating by parts in determining the boundary contribution to the moment maps.)

To get some insight, we look at the space of solutions of Nahm's equations as a complex manifold in one of its complex structures. We introduce $\mathcal{X}_L = X_{L,1+i2}$, $\mathcal{X}_R = X_{R,1+i2}$. Also, from the point of view of one complex structure, the bifundamental hypermultiplet is equivalent to a pair A, B where A is an $n\times m$ matrix and B is an $m\times n$ matrix. The complex moment maps are $\mu_{\mathbb{C},L}=AB$, $\mu_{\mathbb{C},R}=-BA$, and the boundary conditions are therefore

$$\begin{aligned} \mathcal{X}_L(0) &= AB, \\ \mathcal{X}_R(0) &= BA. \end{aligned} \tag{3.80}$$

Of course, Nahm's equations imply that $\mathcal{X}_L(y)$ and $\mathcal{X}_R(y)$ are conjugate for all y to $\mathcal{X}_L(0)$ and $\mathcal{X}_R(0)$.

It follows from (3.80) that the nonzero eigenvalues of $\mathcal{X}_L$ and $\mathcal{X}_R$ are the same. If $n>m$, then $\mathcal{X}_L$ is at most of rank n. If $n=m$, then $\mathcal{X}_L$ and $\mathcal{X}_R$ have the same characteristic polynomials and are conjugate if they are regular, but in general not otherwise.

[13] To preserve the same supersymmetry as that of D3-branes that span directions 0123 and D5-branes that span directions 012456, the NS5-brane should span directions 012789.

We briefly conclude with some examples (which will be useful elsewhere). For $n = 2$ and $m = 1$, (3.80) says that $\mathcal{X}_L(0)$ can be any 2×2 matrix of rank 1, and that $\mathcal{X}_R(0) = \mathrm{Tr}\,\mathcal{X}_L(0)$. Now let us embed this problem in a larger one. We assume that we want to solve Nahm's equations on the interval $(-\ell, \ell]$, by 2×2 matrices for $y < 0$, 1×1 matrices for $y > 0$, and with an NS5-brane at $y = 0$. Also, let us ask for a regular Nahm pole at $y = -\ell$ and Dirichlet boundary conditions at $y = \ell$. All nonzero conjugacy classes arise in the Slodowy slice transverse to a regular Nahm pole, so these boundary conditions allow $\mathcal{X}_L(0)$ to be any nonzero matrix, and in particular any matrix of rank 1. Thus, a solution with the indicated boundary conditions does exist. Since $\mathcal{X}_L(0)$ must be nonzero, the hypermultiplets A and B are likewise nonzero. The group $U(1)$ acts on the space of solutions, by gauge transformations at $y = \ell$. Because A and B are nonzero, a solution with these boundary conditions cannot be $U(1)$-invariant.

Finally, let us consider $n = m = 2$, with the same boundary conditions on the interval $(-\ell, \ell]$, still with a regular Nahm pole at $y = -\ell$, Dirichlet boundary conditions at $y = \ell$, and a fivebrane at $y = 0$. The group that acts on the space of solutions by gauge transformations at $y = \ell$ is now $G = U(2)$. It is possible to find a solution with these boundary conditions that is invariant under a non-central subgroup of $U(2)$ consisting of matrices of the form $\mathrm{diag}(1, *)$. To do this, simply embed the $m = 1$ solution of the last paragraph in the $m = 2$ problem.

4 Supersymmetry without Lorentz Invariance

What were described in Sect. 2 were Lorentz-invariant half-BPS boundary conditions. Here we will discuss what happens if the requirement of Lorentz invariance is dropped. By the Lorentz group in this context we mean $SO(1,2)$, the group of Lorentz transformations that act trivially on $y = x^3$. As we will see, it is possible to break Lorentz invariance but still preserve eight supersymmetries.

Though it is possible to explain this purely in field theory, and we will do so, we will introduce the subject by describing brane constructions that give significant examples. We will simply deform the usual D3-D5 and D3-NS5 systems by turning on a flux on the fivebrane, in a way that "rotates" the unbroken supersymmetries while preserving their number. The deformation breaks both the $SO(1,2)$ Lorentz symmetry and the $SU(2)_X$ R-symmetry, but leaves $SU(2)_Y$ and translation symmetry in the 012 directions.

4.1 Deforming the D3-D5 System

We start with the D3-D5 system in Type IIB superstring theory. This theory in $\mathbb{R}^{1,9}$ has 32 supersymmetries, consisting of two copies ε_L and ε_R of the **16** of $SO(1,9)$. Here ε_L and ε_R arise respectively from left- and right-moving excitations on the string worldsheet. Now as usual we introduce four-dimensional $U(N)$ gauge theory by considering N D3-branes with worldvolume extending in the directions 0123. Half of the supersymmetry is broken; the unbroken supersymmetries obey

$$\varepsilon_R = \Gamma_{0123}\varepsilon_L = -B_0\varepsilon_L. \tag{4.1}$$

(The B_i were defined in (2.7).) Then we introduce a D5-brane extending in directions 012456. This again reduces the supersymmetry by a factor of two; in the absence of any flux, the unbroken supersymmetries obey $\varepsilon_R = \Gamma_{012456}\varepsilon_L = -B_0B_1\varepsilon_L$.

The D5-brane supports a $U(1)$ gauge field, whose curvature we will call $\boldsymbol{F}$ and measure in string units. We take $\boldsymbol{F}$ to be a two-form with constant coefficients on the D5-brane worldvolume, preserving translation invariance but breaking Lorentz invariance. The condition for unbroken supersymmetry due to the presence of the D5-brane is deformed to

$$\varepsilon_R = -\exp(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)B_0B_1\varepsilon_L. \tag{4.2}$$

Generically, the two conditions (4.1) and (4.2) are inconsistent and there are no unbroken supersymmetries. Indeed, we can combine the two equations into

$$\varepsilon_L = -B_0 w B_0 B_1 \varepsilon_L \tag{4.3}$$

where $w=\exp(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)$. We can think of $W=-B_0wB_0B_1$ as an element of $Spin(1,6)$, the double cover of $SO(1,6)$, the Lorentz group acting on directions 0123456. For generic $\boldsymbol{F}$, 1 is not an eigenvalue of W (acting on spinors) and there are no unbroken supersymmetries. For $\boldsymbol{F}=0$, $W=B_1$, which is the lift to $Spin(1,6)$ of the $SO(1,6)$ element $\mathrm{diag}(1,1,1,-1,-1,-1,-1)$. Equation (4.3) then has an eight-dimensional space of solutions. In general, for W to preserve one-half of the supersymmetries, it must belong to an $SU(2)$ subgroup of $Spin(1,6)$ (embedded via $SU(2)\subset SU(2)\times SU(2)=Spin(4)\subset Spin(1,6)$). On the other hand, the explicit form of W shows that it anticommutes with Γ_3, so one of its eigenvalues in the **7** of $SO(1,6)$ is -1. So W must be conjugate to $\mathrm{diag}(1,1,1,-1,-1,-1,-1)$. In particular, $W^2=1$, which is equivalent to

$$B_1 w B_1 = w^{-1}. \tag{4.4}$$

That equation holds if

$$B_1\boldsymbol{F} = -\boldsymbol{F}B_1. \tag{4.5}$$

Conversely, (4.5) gives a component of solutions of (4.4), namely the component of solutions that come by deformation from $\boldsymbol{F}=0$. We will only consider this component.

For $\boldsymbol{F}$ of the form found in the last paragraph, $B_0w=w^{-1}B_0$, so the condition (4.3) for unbroken supersymmetry simplifies to $\varepsilon_L=w^{-1}B_1\varepsilon_L$. If we introduce a natural square root of w by

$$h=\exp(\Gamma^{IJ}\boldsymbol{F}_{IJ}/8), \tag{4.6}$$

the condition becomes

$$h^{-1}B_1 h\varepsilon_L = \varepsilon_L. \tag{4.7}$$

The matrix h is an element of $SO(1,5)$ (acting on directions 012456), and the eight-dimensional subspace of unbroken supercharges is conjugate under h^{-1} to the standard space of unbroken supersymmetries of the D3-D5 system.

4.1.1 Field Theory Interpretation

We will now reinterpret purely in field theory terms the brane construction just described. One advantage of this is that the field theory description is valid for arbitrary gauge group, not just for gauge groups (such as $U(N)$) that are conveniently realized by branes.

So far we could be considering intersecting branes or D3-branes ending on a D5-brane. We now focus on that latter case.

At $\boldsymbol{F}=0$, the appropriate boundary condition for N D3-branes ending on a D5-brane was described in Sect. 3.4.3. $\vec{Y}$ obeys Dirichlet boundary conditions, as do the three-dimensional gauge fields A_μ. However, the scalar fields X^a do not obey simple Dirichlet or Neumann boundary conditions, but have a pole at the boundary:

$$X^a \sim \frac{t^a}{y} + \cdots . \tag{4.8}$$

Here $y = x^3$ vanishes at the boundary and t^a are the images of standard $\mathfrak{su}(2)$ generators for a principal embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. The unbroken supersymmetries obey

$$B_1 \varepsilon = \varepsilon. \tag{4.9}$$

When we turn on $\boldsymbol{F}$, the D5-brane is still located at $\vec{Y}=0$, so there is no change in the Dirichlet boundary conditions for $\vec{Y}$. However, the boundary conditions on A_μ and $\vec{X}$ do change.

The boundary condition (4.8) is supersymmetric because the polar behavior of the X^a is consistent with Nahm's equations

$$\frac{DX^a}{Dy} + \frac{1}{2}\epsilon^{abc}[X_b, X_c] = 0. \tag{4.10}$$

This equation is consistent with supersymmetry because Nahm's equations are the dimensional reduction of the selfdual Yang-Mills equations, which are of course compatible with supersymmetry. Here we are considering the selfdual Yang-Mills equations in the 3456 plane, even though the covariant derivatives $D_a = \partial_a + A_a$ in the 456 direction have been replaced by matrices X^a.

The selfdual Yang-Mills equations in any four-dimensional plane preserve the same amount of supersymmetry. Therefore, we can make an $SO(1,5)$ rotation of the polar behavior that is assumed in (4.10). (Of course, we use the $SO(1,5)$ that fixes the 3 direction and acts on 012456.) We take three orthonormal linear combinations of the 012456 directions, and postulate that the corresponding fields C^i (which are orthonormal linear combinations of A_μ, $\mu = 0,1,2$ and X^a, $a = 4,5,6$) have a pole $C^i \sim t^i/y$. This preserves supersymmetry, just like the special case of (4.8), since it is a special solution of the selfdual Yang-Mills equations (or a dimensional reduction thereof) in a certain four-dimensional subspace.

We thus get a family of boundary conditions that are all associated with a principal embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. They are obtained by making an $SO(1,5)$ rotation of the pole at $y=0$, even though $SO(1,5)$ is not a symmetry of the theory. The unbroken supersymmetry is obtained by making the same $SO(1,5)$ rotation from the 3456 plane to the appropriate four-plane.[14] Thus, we can immediately characterize the supersymmetry left unbroken by a boundary condition of this type. For some element $h \in SO(1,5)$, the unbroken supersymmetries obey the rotated version of (4.9), namely

$$h^{-1} B_1 h \varepsilon = \varepsilon \tag{4.11}$$

or

$$B_1 h \varepsilon = h \varepsilon. \tag{4.12}$$

[14] The relevant rotation group is $SO(1,5)$, not $SO(1,6)$, because we do not rotate the $y = x^3$ direction. This direction is distinguished by the fact that the boundary is at $y=0$.

This coincides with the condition (4.7) that we found for the D3-D5 system (in the present field theory approach, we denote ε_L simply as ε).

It is convenient to also introduce the six-dimensional chirality operator $\Gamma' = \Gamma_{012456}$ and make a chiral decomposition $\varepsilon = \varepsilon_+ + \varepsilon_-$, where

$$\Gamma'\varepsilon_\pm = \pm\varepsilon_\pm. \tag{4.13}$$

Since B_1 anticommutes with Γ', (4.12) is equivalent to

$$h\varepsilon_- = B_1 h\varepsilon_+. \tag{4.14}$$

Because of (4.13), we can replace $B_1 = \Gamma_{3456}$ by $\Gamma^* = \Gamma_{0123}$ and write

$$h\varepsilon_- = -\Gamma^* h\varepsilon_+. \tag{4.15}$$

This will be useful in Sect. 4.3.

The condition (4.15) is invariant under $h \to qh$ for $q \in Q = SO(1,2) \times SO(3)_X$ (which commutes with Γ^*). So we can think of h as taking values in the nine-dimensional space $\tilde{Z}_9 = Q\backslash SO(1,5)$. (Of course, nine is also the number of components of $\boldsymbol{F}$, given that it has one index of type 012 and one of type 456.) Thus, this construction gives a nine-dimensional family $\tilde{Z}_9$ of half-BPS boundary conditions that are associated with the principal embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. This generalizes what we found from the D3-D5 system for $G = U(N)$.

4.2 Rotating the D3-NS5 System

Similarly it is possible to "rotate" the unbroken supersymmetry of the D3-NS5 system. This is particularly simple if the four-dimensional θ-angle vanishes.

We first rewrite (4.2) in the form

$$\begin{pmatrix} \varepsilon_R \\ \varepsilon_L \end{pmatrix} = -\exp\left(\frac{1}{4}\Gamma^{IJ}\boldsymbol{F}_{IJ}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) B_0 B_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_R \\ \varepsilon_L \end{pmatrix}. \tag{4.16}$$

At $\theta = 0$, the duality transformation $S : \tau \to -1/\tau$ maps a D5-brane to an NS5-brane and transforms ε_R, ε_L to

$$\begin{pmatrix} \varepsilon'_R \\ \varepsilon'_L \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon_R \\ \varepsilon_L \end{pmatrix}. \tag{4.17}$$

The supersymmetry condition in the presence of an NS5-brane can be deduced from (4.16) and is

$$\begin{pmatrix} \varepsilon'_R \\ \varepsilon'_L \end{pmatrix} = -\exp\left(-\frac{1}{4}\Gamma^{IJ}\boldsymbol{F}_{IJ}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) B_0 B_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \varepsilon'_R \\ \varepsilon'_L \end{pmatrix}. \tag{4.18}$$

If also D3-branes are present, we must supplement this with

$$\varepsilon'_R = -B_0\varepsilon'_L, \tag{4.19}$$

which follows from (4.1). With a little algebra, one can eliminate ε'_R and obtain the condition on ε'_L:

$$(1 - \cosh(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)B_2 + \sinh(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)B_1)\varepsilon'_L = 0. \tag{4.20}$$

(In deriving this, note that $\Gamma^{IJ}\boldsymbol{F}_{IJ}$ commutes with B_2, and anticommutes with B_0 and B_1.)

As preparation for interpreting this result in field theory, we again make a chiral decomposition $\varepsilon'_L = \varepsilon_+ + \varepsilon_-$, where

$$\Gamma' \varepsilon_\pm = \pm \varepsilon_\pm. \tag{4.21}$$

(In field theory, we omit the primes and the subscript L and denote the supersymmetry generator simply as ε.) We can use (4.20) to solve for ε_- in terms of ε_+:

$$\varepsilon_- = \frac{1}{1-\cosh(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)B_2}\sinh(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)B_1\varepsilon_+. \tag{4.22}$$

We can make a small simplification as follows. We have $B_2\varepsilon_+ = -\varepsilon_+$ (since $\Gamma'\varepsilon_+ = \varepsilon_+$ and ε_+ has positive ten-dimensional chirality). Also B_2 commutes with $\Gamma^{IJ}\boldsymbol{F}_{IJ}$ and anticommutes with B_1. Using these facts, one can omit B_2 in (4.22), which becomes

$$\varepsilon_- = \frac{1}{1-\cosh(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)}\sinh(\Gamma^{IJ}\boldsymbol{F}_{IJ}/4)B_1\varepsilon_+. \tag{4.23}$$

Expanding this in powers of $\boldsymbol{F}$, the first term is

$$\varepsilon_- = \frac{1}{4}\Gamma^{IJ}\boldsymbol{F}_{IJ}B_1\varepsilon_+. \tag{4.24}$$

Using the fact that explicitly $B_1 = \Gamma_{3456}$, and that $\boldsymbol{F}$ has one index of type 012 and one of type 456, we see we can write this as

$$\varepsilon_- = \sum_{I<J<K}\Gamma^{IJK}q_{IJK}\Gamma_3\varepsilon_+, \tag{4.25}$$

where here the indices I, J, K take values 012456, and q_{IJK} is a third rank antisymmetric tensor that depends on $\boldsymbol{F}$.

It is convenient to regard $q = \sum_{I<J<K} q_{IJK}\mathrm{d}x^I \wedge \mathrm{d}x^J \wedge \mathrm{d}x^K$ as a three-form on $\mathbb{R}^{1,5}$ with constant coefficients. It is not immediately obvious that q is selfdual or anti-selfdual,[15] but in fact, because $\Gamma'\varepsilon_+ = \varepsilon_+$, the anti-selfdual part of q does not contribute, and hence we can project q to its selfdual part. Let $\star_{012}$ and $\star_{456}$ be the Hodge $\star$ operators in the 012 and 456 directions. We can pick conventions so that $\star_{012}^2 = \star_{456}^2 = 1$, $\star_{012}\star_{456} = \star_{456}\star_{012}$; the six-dimensional $\star$ operator is $\star = \star_{012}\star_{456}$. The relation between q and $\boldsymbol{F}$ in linear order is

$$q = \frac{(\star_{012} + \star_{456})\boldsymbol{F}}{8} \tag{4.26}$$

and here q is selfdual.

Though this analysis has been only to linear order in $\boldsymbol{F}$, in fact, (4.23) is precisely equivalent to (4.25), with the selfdual three-form q in general a nonlinear function of $\boldsymbol{F}$. To see this, we observe that gamma matrices $\Gamma_7, \Gamma_8, \Gamma_9$ are absent in (4.23) and Γ_3 appears only as a linear factor in B_1 multiplying ε_+. So (4.23) takes the form $\varepsilon_- = \Omega\Gamma_3\varepsilon_+$, where Ω is constructed from gamma matrices Γ^I, with I ranging over 012456. Ω must be of odd order

[15] We define an antisymmetric tensor ϵ_{IJKLMN} with $\epsilon^{012456} = 1$. Indices I, J, K will take values $0, 1, 2, 4, 5, 6$. Self-duality for a third rank antisymmetric tensor q means that $q^{IJK} = \epsilon^{IJKLMN}q_{LMN}/3!$. In Lorentz signature in six dimensions, a third rank real antisymmetric tensor can be selfdual or anti-selfdual. For example, with this definition, the three-form $-\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^4 \wedge \mathrm{d}x^5 \wedge \mathrm{d}x^6$ is selfdual.

in the Γ^I, since it must reverse the six-dimensional chirality; and because ε_+ obeys (4.21), we can reduce to the case $\Omega = \Gamma^I S_I + \sum_{I<J<K} \Gamma^{IJK} q_{IJK}$, with a one-form S and selfdual three-form q. Moreover, the one-form is absent for a reason that will be explained in Sect. 4.4.

Thus, the unbroken supersymmetry can be characterized by a selfdual three-form in six dimensions. However, the construction as described so far does not lead to the most general selfdual three-form. Indeed, as in Sect. 4.1, $\boldsymbol{F}$ depends on only nine parameters, but a selfdual three-form (with constant coefficients) in $\mathbb{R}^{1,5}$ depends on ten parameters. The missing parameter is the four-dimensional θ angle, which preserves half of the supersymmetry (and actually preserves Lorentz invariance). It is absent from the above formulas because we obtained them starting with S-duality from the D3-D5 system at $\theta=0$. This tenth parameter will be included in Sect. 4.2.1 as well as Sect. 4.4.

If we restrict to $\theta=0$, we get a nine-parameter family Z_9 of half-BPS (but not Lorentz-invariant) deformations of the D3-NS5 system. They are S-dual to the corresponding nine-parameter family $\tilde{Z}_9$ of deformations of the D3-D5 system, described in Sect. 4.1, in the sense that the strong coupling limit of one is the weak coupling limit of the other.

The reason that we have not seen a tenth parameter for the D3-D5 system is that S-duality becomes more complicated when $\theta \neq 0$; it does not simply exchange weak and strong coupling. As soon as $\theta \neq 0$, the S-dual of a strongly coupled D3-NS5 system is no longer a weakly coupled D3-D5 system.

4.2.1 Realization in Field Theory

We will now re-examine the deformation of the D3-NS5 system just described from the point of view of field theory. As usual, one advantage of this is that the discussion is valid for any gauge group.

D3-branes ending on a single NS5-brane without any flux are governed by Neumann boundary conditions for the vector multiplet A_μ and $\vec{X}$ and Dirichlet boundary conditions for $\vec{Y}$. This was described in Sect. 2. Can we modify these boundary conditions in a way that depends on a selfdual or anti-selfdual third rank tensor and preserves the half-BPS property? In fact, in a special case this has essentially been done in Sect. 2.

In that analysis, a deformation was considered from Neumann boundary conditions for gauge fields, which assert that $F_{3\lambda}=0$ on the boundary for $\lambda=0,1,2$, to a more general boundary condition with three-dimensional Lorentz invariance:

$$\epsilon_{\lambda\mu\nu} F^{3\lambda} + \gamma F_{\mu\nu} = 0. \tag{4.27}$$

The physical meaning of the term linear in γ was explained in (2.20). It corresponds to adding to the action a term proportional to $\int \operatorname{Tr} F \wedge F$, or equivalently, after integrating by parts to convert this to a surface term, it corresponds to adding a boundary interaction

$$-\frac{\gamma}{2e^2}\int_{\partial M} \mathrm{d}^3x \epsilon^{\mu\nu\lambda} \operatorname{Tr}\left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda\right). \tag{4.28}$$

Here M is spacetime, and ∂M is its boundary at $y=0$. The interaction that we have added is of dimension three and therefore preserves conformal invariance.

$SO(1,2)$ invariance allows us to add one more conformally-invariant interaction constructed from bosons. This is

$$-\frac{u}{3e^2}\int_{\partial M} \mathrm{d}^3x \epsilon_{abc} \operatorname{Tr} X^a [X^b, X^c]. \tag{4.29}$$

If we do add this interaction, then Neumann boundary conditions for X are modified to

$$\frac{DX_a}{Dy} + \frac{u}{2}\epsilon_{abc}[X_b, X_c] = 0. \tag{4.30}$$

If we are willing to relax $SO(1,2)$ invariance, we can add additional bosonic interactions that preserve global scale invariance. Define quantities Z^I, $I = 0,1,2,4,5,6$, as follows. For $I = 4,5,6$, set $Z^I = X^a$. And for $I = 0,1,2$, set Z^I equal to the covariant derivative $D_I = \partial_I + A_I$. Then we can add to the action a dimension three term that we loosely describe as

$$\int_{\partial M} \mathrm{d}^3x q_{IJK} \operatorname{Tr} Z^I[Z^J, Z^K], \tag{4.31}$$

where q is an arbitrary third-rank antisymmetric tensor. The special case involving the component q_{012} corresponds to the Chern-Simons interaction in (4.28), and the case involving q_{456} corresponds to the Lorentz-invariant coupling in (4.29). Other components of q give couplings that violate Lorentz invariance; they are schematically of the form $\operatorname{Tr} X^a F_{\mu\nu}$ or $\operatorname{Tr} X^a D_\mu X^b$, with $\mu, \nu = 0,1,2$, $a, b = 4,5,6$.

Now the question arises of whether the bosonic interaction (4.31) can be completed to a supersymmetric theory by suitably modifying the fermion boundary conditions (or equivalently, by adding boundary interactions bilinear in fermions). If so, will a constraint come in related to selfduality or anti-selfduality? We would expect this from the discussion of (4.25).

Happily, we do not really need to do a new calculation. For the Lorentz-invariant case, with q_{012} and q_{456} the only non-zero matrix elements of q, a half-BPS boundary condition was constructed in Sect. 2.1. The quantities γ and u were not independent but were parametrized by

$$\gamma = -\frac{2a}{1-a^2}, \qquad u = -\frac{2a}{1+a^2}. \tag{4.32}$$

NS5-brane and NS5-antibrane boundary conditions correspond to $a = \infty$ and $a = 0$. Expanding to first order in $1/a$ near $a = \infty$ or to first order in a near $a = 0$, we have $\gamma = \mp u$, which corresponds to the expected selfduality or anti-selfduality of the tensor q. The condition $\gamma = \mp u$ means that the three-form q is

$$q = u(\mp \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^4 \wedge \mathrm{d}x^5 \wedge \mathrm{d}x^6) \tag{4.33}$$

and so is Lorentz-invariant. Note that this particular three-form cannot be expressed in terms of $\boldsymbol{F}$ as in (4.26), so we are here indeed describing the tenth parameter that was missing in that derivation.

4.2.2 Canonical Form of q

One might think that the supersymmetry of the construction of Sect. 2 that we have just reviewed is only a special case. But in a certain sense it is actually generic. Let us count the number of parameters of a general selfdual three-form that, by an $SO(1,5)$ transformation, can be put in the form of (4.33). One parameter, namely u, is visible in (4.33). We must also allow 9 more parameters generated by $SO(1,5)$ transformations. ($SO(1,5)$ has dimension 15; its subgroup that leaves q fixed is $SO(1,2) \times SO(3)$, of dimension 6; the difference is 9.) This gives a total of $1 + 9 = 10$ parameters. But 10 is the dimension of the space of selfdual or anti-selfdual three-forms, so a generic such form is of this type.

The half-BPS boundary condition derived from D3-branes ending on an NS5-brane actually has a direct analog in $6+1$-dimensional super Yang-Mills theory. In string theory, this can be understood by replacing the D3-branes ending on an NS5-brane by D6-branes which end on the NS5-brane.[16] From a field theory point of view, we simply allow all fields to depend on three more coordinates x^4, x^5, x^6, and replace the three scalar fields X^a with covariant derivatives $D_a + A_a$ in the x^4, x^5, x^6 directions. This substitution makes sense because X^a enters the $\mathcal{N}=4$ super Yang-Mills Lagrangian only via its commutators with other fields and with covariant derivatives.

The boundary condition for $D6$-branes ending on an NS5-brane has $SO(1,5)$ symmetry. So after lifting the D3-NS5 system to $6+1$ dimensions, and making the deformation involving the three-form in (4.33), we can make an $SO(1,5)$ rotation. Then we can reduce back to $3+1$ dimensions, taking the fields to be once again independent of x^4, x^5, x^6, and turning the covariant derivatives $D_a + A_a$ back into scalar fields X^a.

What we gain by the detour through $6+1$ dimensions is the knowledge that we can, in effect, make an $SO(1,5)$ transformation of the deformed boundary conditions even though $SO(1,5)$ is not a symmetry of the theory. Hence, without any need for further computation, there is a half-BPS boundary condition in which (4.33) is replaced by a general selfdual three-form.

This construction gives a ten-dimensional family Z_{10} of half-BPS boundary conditions. The Neumann boundary conditions of the D3-NS5 system, with any number of D3-branes and a single NS5-brane, represent a point in Z_{10}. The generic point represents a half-BPS but not Lorentz-invariant deformation. At a generic point, the unbroken supersymmetry is described by

$$\varepsilon_- = \sum_{I<J<K} \Gamma^{IJK} q_{IJK} \Gamma_3 \varepsilon_+ \tag{4.34}$$

for a selfdual three-form q. As we explain in Sect. 4.4, the family Z_{10} also contains points "at infinity" that cannot be described in this way. (These include points describing D3-branes ending on an NS anti-fivebrane.)

Only the sublocus Z_9 describes deformations that have a simple S-duality relationship to the analogous family $\tilde{Z}_9$ of deformations of the D3-D5 system. For a given q, how can we determine if the corresponding deformation of the D3-NS5 system lies in Z_9? One necessary and sufficient criterion is that it must be possible to parametrize q via (4.23) in terms of a two-form $\boldsymbol{F}$. An equivalent criterion is that the space of unbroken supersymmetries, characterized by (4.34), must transform under $S: \tau \to -1/\tau$ into a space of supersymmetries that can be characterized in terms of the analogous formula (4.11) of the D3-D5 system.

To use the last-mentioned criterion, we need to know how the space of unbroken supersymmetries transforms under duality. This can be deduced from string theory formulas presented earlier, but can also be understood purely in four-dimensional terms. In general, under a duality transformation that transforms the coupling parameter τ by $\tau \to (a\tau+b)/(c\tau+d)$, the supersymmetry generators ε transform by

$$\varepsilon \to \left(\frac{|c\tau+d|}{c\tau+d}\right)^{-i\Gamma^*/2} \varepsilon, \tag{4.35}$$

[16] Since the NS5-brane, which is supposed to provide the boundary, has a six-dimensional world-volume, we cannot make a construction like this above $6+1$ dimensions. This can also be understood from a field theory point of view; the Dirichlet boundary conditions on the three scalar fields Y^p do not have analogs if one or more of those scalars is replaced by covariant derivatives in extra dimensions.

with $\Gamma^* = \Gamma_{0123}$. (For example, see [19], (2.25).) For the transformation $S : \tau \to -1/\tau$, with $\theta = 0$ so that τ is on the imaginary axis, this becomes

$$\varepsilon \to \frac{1-\Gamma^*}{\sqrt{2}}\varepsilon. \tag{4.36}$$

4.3 An Example

Now we are going to consider an example: we will take a boundary condition representing a point in Z_{10}, and show that it actually represents a point in Z_9, and so is S-dual to a D3-D5 boundary condition with a pole.

As explained in Sect. 4.2.2, a generic selfdual three-form q can be put in the canonical form of (4.33). But it is not true that every selfdual three-form can be put in this form. A counterexample can be written

$$q = \frac{1}{4}(\mathrm{d}x^0 + \mathrm{d}x^4) \wedge (\mathrm{d}x^1 \wedge \mathrm{d}x^5 + \mathrm{d}x^2 \wedge \mathrm{d}x^6). \tag{4.37}$$

This three-form cannot be put in the form (4.33) by an $SO(1,5)$ transformation, because when that is done, $|u|$ is an invariant. However, q can be rescaled by a Lorentz boost in the 04 plane, and hence cannot be characterized by any nonzero invariant.

A deformation of the D3-NS5 system associated with this choice of q appears in the gauge theory approach to geometric Langlands.[17] The fact that the S-dual of this particular boundary condition is associated with a point in $\tilde{Z}_9$, and thus is associated with a principal embedding $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, is important in geometric Langlands, and has until now been mysterious from the gauge theory point of view.

We can relate this particular deformation of the D3-NS5 system to a D3-D5 deformation using either of the two approaches mentioned at the end of Sect. 4.2.2. First, we can show directly that with q as above, the deformed NS5 supersymmetry relation $\varepsilon_- = \sum_{I<J<K} \Gamma^{IJK} q_{IJK} \Gamma_3 \varepsilon_+$ is equivalent to the deformed D5 relation (4.23), with $\boldsymbol{F}$ a multiple of $\mathrm{d}x^1 \wedge \mathrm{d}x^6 - \mathrm{d}x^2 \wedge \mathrm{d}x^5$. (The precise multiple is determined below by another method.) The evaluation of (4.23) is simple for $\boldsymbol{F}$ of this form because $M = \Gamma_{16} - \Gamma_{25}$ obeys $M^3 = -4M$, reflecting the fact that it is a generator of an $SU(2)$ subgroup of $Spin(1,5)$.

Alternatively, we can proceed by analyzing the unbroken supersymmetries. As usual, we write the generator of an unbroken supersymmetry as $\varepsilon = \varepsilon_+ + \varepsilon_-$, where

$$\Gamma' \varepsilon_\pm = \pm \varepsilon_\pm, \tag{4.38}$$

and moreover

$$\varepsilon_- = \sum_{I<J<K} q_{IJK} \Gamma^{3IJK} \varepsilon_+ = \frac{1}{4}\Gamma_3(-\Gamma_0 + \Gamma_4)(\Gamma_{15} + \Gamma_{26})\varepsilon_+. \tag{4.39}$$

[17] In (12.31) of [19], boundary conditions are given for a gauge theory description of the "canonical coisotropic brane." These boundary conditions can be obtained by perturbing $\mathcal{N} = 4$ super Yang-Mills by a boundary interaction associated with a three-form, as in (4.31). The necessary three-form is the one indicated in (4.37). To see this, one must take into account a clash in notation between the present paper and [19]. The boundary direction that we call x^3 is called x^1 in [19], and the directions that we label 012456 are 023567 in [19].

According to (4.36), the duality transformation $S:\tau\to -1/\tau$ maps ε to $\tilde{\varepsilon}=\frac{1}{\sqrt{2}}(1-\Gamma^*)\varepsilon$, or equivalently $\varepsilon=\frac{1}{\sqrt{2}}(1+\Gamma^*)\tilde{\varepsilon}$. Since Γ^* anticommutes with Γ', it exchanges $\varepsilon_\pm$, so this becomes

$$\begin{aligned}\varepsilon_+ &= \frac{1}{\sqrt{2}}(\tilde{\varepsilon}_+ + \Gamma^*\tilde{\varepsilon}_-),\\ \varepsilon_- &= \frac{1}{\sqrt{2}}(\tilde{\varepsilon}_- + \Gamma^*\tilde{\varepsilon}_+).\end{aligned} \tag{4.40}$$

If we set $M=\frac{1}{4}\Gamma_3(-\Gamma_0+\Gamma_4)(\Gamma_{15}+\Gamma_{26})$, so that (4.39) reads $\varepsilon_-=M\varepsilon_+$, then the S-dual version is $\tilde{\varepsilon}_-+\Gamma^*\tilde{\varepsilon}_+=M(\tilde{\varepsilon}_++\Gamma^*\tilde{\varepsilon}_-)$, or

$$(1-M\Gamma^*)\tilde{\varepsilon}_- = -\Gamma^*(1+\Gamma^*M)\tilde{\varepsilon}_+. \tag{4.41}$$

Upon evaluating $M\Gamma^*\tilde{\varepsilon}_-$ and $\Gamma^*M\tilde{\varepsilon}_+$, using (4.38), we find after some gamma matrix algebra that (4.41) is equivalent to

$$P\tilde{\varepsilon}_- = -\Gamma^* P\tilde{\varepsilon}_+, \tag{4.42}$$

where

$$P = 1+\frac{\Gamma_{16}-\Gamma_{25}}{2}. \tag{4.43}$$

If P were an element of $SO(1,5)$, this relation would be in the desired form (4.15), showing that the S-dual of the boundary condition that we started with does represent a point in $\tilde{Z}_9$. It is actually not true that P is an element of $SO(1,5)$. However, both P and Γ^* commute with

$$T = \frac{1-\Gamma_{1256}}{2}+\frac{1+\Gamma_{1256}}{2\sqrt{2}}, \tag{4.44}$$

so (4.43) is equivalent to

$$TP\tilde{\varepsilon}_- = -\Gamma^* TP\tilde{\varepsilon}_+. \tag{4.45}$$

Here[18]

$$TP = \left(\frac{1-\Gamma_{1256}}{2}+\frac{1+\Gamma_{1256}}{2\sqrt{2}}\right)\left(1+\frac{\Gamma_{16}-\Gamma_{25}}{2}\right) = \exp\left(\frac{\pi}{8}(\Gamma_{16}-\Gamma_{25})\right) \tag{4.46}$$

is an element of $SO(1,5)$, in fact an element of the one-parameter subgroup of $SO(1,5)$ generated by $\Gamma_{16}-\Gamma_{25}$. So (4.45) is of the form of (4.15).

4.4 General Formulation

Until this point, we have relied upon explicit constructions using either branes or field theory. Here, we will study conceivable half-BPS boundary conditions from a more general point of view. This will give a clearer understanding of some things that we originally described by hand.

[18] To obtain the second equality in (4.46), observe that both sides equal 1 when acting on spinors ψ with $\Gamma_{1256}\psi=-\psi$, and then evaluate the two sides assuming $\Gamma_{1256}\psi=\psi$.

First of all, if one has a boundary at $x^3 = 0$, then regardless of the nature of the boundary condition, there is no translation invariance in the x^3 direction. Hence if ε and $\tilde{\varepsilon}$ are two generators of supersymmetries that remain valid in the presence of the boundary, we must have

$$\overline{\varepsilon}\Gamma_3\tilde{\varepsilon} = 0. \tag{4.47}$$

For a half-BPS boundary condition, 8 of the possible 16 supersymmetries are unbroken. We can interpret the condition (4.47) as follows. Let V_{16} be the 16-dimensional real vector space (the irreducible positive chirality spinor representation of $SO(1,9)$) in which ε takes values. The expression $(\varepsilon, \tilde{\varepsilon}) = \overline{\varepsilon}\Gamma_3\tilde{\varepsilon}$ defines a non-degenerate quadratic form on this space, of signature $(8,8)$. The condition (4.47) asserts that ε takes values in a subspace $T \subset V_{16}$ such that the quadratic form vanishes when restricted to T. A maximal subspace with this property is eight-dimensional, and the half-BPS condition asserts precisely that T is maximal.

Regardless of what we pick T to be, the condition that $\varepsilon, \tilde{\varepsilon} \in T$ does not suffice to set $\overline{\varepsilon}\Gamma^\mu\tilde{\varepsilon} = 0$ for any value of μ other than 3. Hence, a half-BPS boundary condition, though not necessarily Lorentz-invariant, is invariant under translations in the 0, 1, and 2 directions.

Let $\mathcal{S}$ be the space of all eight-dimensional null subspaces of V_{16}. Every half-BPS boundary condition determines a point in $\mathcal{S}$. $\mathcal{S}$ is a homogeneous space for a group $H = SO(1,8)$ that formally rotates the coordinates x^I, $I \neq 3$. $SO(1,8)$ is not really a symmetry group of $\mathcal{N} = 4$ super Yang-Mills theory; only its subgroup $SO(1,2) \times SO(6)$ is a group of symmetries. ($SO(1,2)$ is the Lorentz group that acts on the 0, 1 and 2 directions, and $SO(6)$ is the group of R-symmetries.) But the action of $SO(1,8)$ on $\mathcal{S}$ will be useful in the following analysis of half-BPS boundary conditions that lack $SO(1,2)$ symmetry.

We make a preliminary simplification along the following lines. We will only consider half-BPS boundary conditions that can be obtained by marginal (scale-invariant) deformation of a Lorentz-invariant one. As explained in Sect. 2.1, any $SO(1,2)$-invariant half-BPS boundary condition has Dirichlet boundary conditions on precisely three of the scalars, denoted there as $\vec{Y}$. Since the only fields in $\mathcal{N} = 4$ super Yang-Mills theory of conformal dimension 1 are the scalar fields $\vec{X}$ and $\vec{Y}$, the only possible marginal deformation of the Dirichlet boundary condition $\vec{Y}| = 0$ is to rotate $\vec{Y}$ to a linear combination of $\vec{X}$ and $\vec{Y}$. Making such a rotation does not give anything essentially new, so we will stick with $\vec{Y}| = 0$.

Furthermore, we will consider only half-BPS boundary conditions that are invariant under the group $SO(3)_Y$ that rotates $\vec{Y}$. It is now useful to decompose the space V_{16} under the action of $SO(1,5) \times SO(3)_Y$, where $SO(1,5)$, which rotates the directions 012456, is the subgroup of $SO(1,8)$ that commutes with $SO(3)_Y$. We can decompose V_{16} as $W_8 \otimes W_2$, where $SO(3)_Y$ acts on W_2 in the spinor representation, and $SO(1,5)$ likewise acts on W_8 in the spinor representation. (Both $SO(1,5)$ chiralities are included in W_8.) For $\varepsilon = \mu \otimes \nu$, $\tilde{\varepsilon} = \tilde{\mu} \otimes \tilde{\nu}$, we can decompose the inner product as

$$(\varepsilon, \tilde{\varepsilon}) = \langle \mu, \tilde{\mu} \rangle \langle \nu, \tilde{\nu} \rangle', \tag{4.48}$$

where $\langle\ ,\ \rangle$ is an inner product on W_8 and $\langle\ ,\ \rangle'$ is one on W_2. The second inner product $\langle\ ,\ \rangle'$ is antisymmetric (the spinor representation of $SO(3)_Y$ admits only an antisymmetric inner product), so $\langle\ ,\ \rangle$ is also antisymmetric.[19] The decomposition $V_{16} = W_8 \otimes W_2$ is obviously

[19] This does not follow from $SO(1,5)$ invariance alone; since W_8 is the direct sum of the two spinor representations of $SO(1,5)$ of opposite chirality, it admits both a symmetric and an antisymmetric invariant inner product. This is clear from the group theory described below.

similar to a decomposition made in Sect. 2.1, but here we make this decomposition using a different subgroup of $SO(1,8)$.

Now let us return to the eight-dimensional null subspace $T \subset V_{16}$ that parametrizes the supersymmetries left unbroken by a half-BPS boundary condition. If the boundary condition is to be $SO(3)_Y$-invariant, we must have $T = U \otimes W_2$, where U is a four-dimensional null subspace of W_8.

It will help to know something about such null subspaces. For this, we need some $SO(1,5)$ group theory. Let us write $\mathbf{4}$ and $\mathbf{4}'$ for the positive and negative chirality spinor representations of $SO(1,5)$. Thus, we have $W_8 \cong W_{\mathbf{4}} \oplus W_{\mathbf{4}'}$, where $W_{\mathbf{4}}$ and $W_{\mathbf{4}'}$ transform, respectively, in the representations $\mathbf{4}$ and $\mathbf{4}'$. We denote the trivial representation, the vector representation, and the second rank antisymmetric tensor representation of $SO(1,5)$ as $\mathbf{1}, \mathbf{6}$, and $\mathbf{15}$, respectively. Finally, the third rank tensor representation has dimension $6 \cdot 5 \cdot 4/3! = 20$, but decomposes as a direct sum of two representations $\mathbf{10}$ and $\mathbf{10}'$ that consist respectively of anti-selfdual and selfdual third rank tensors.

The tensor products of spinor representations of $SO(1,5)$ decompose as follows:

$$\begin{aligned} \mathbf{4} \otimes \mathbf{4} &= \mathbf{6}_A \oplus \mathbf{10}_S, \\ \mathbf{4}' \otimes \mathbf{4}' &= \mathbf{6}_A \oplus \mathbf{10}'_S, \\ \mathbf{4} \otimes \mathbf{4}' &= \mathbf{1} \oplus \mathbf{15}. \end{aligned} \tag{4.49}$$

The subscripts A and S refer respectively to the antisymmetric and symmetric parts. For example, the first line means that the antisymmetric part of $\mathbf{4} \otimes \mathbf{4}$ is $\mathbf{6}$ and the symmetric part is $\mathbf{10}$.

From (4.49), we see that an invariant inner product between two spinors must pair a $\mathbf{4}$ and a $\mathbf{4}'$. So $W_{\mathbf{4}}$ and $W_{\mathbf{4}'}$ are two examples of null subspaces of V_8. It is not hard to describe half-BPS boundary conditions associated with these subspaces. The condition that $\mu \in W_4$ is that $\Gamma'\mu = \mu$, where we set $\Gamma' = \Gamma_{012456} = \epsilon^{IJKLMN}\Gamma_{IJKLMN}/6!$. We can write the condition on μ in terms of $\varepsilon = \mu \otimes \nu$ as $\Gamma'\varepsilon = \varepsilon$. Equivalently, since ε has positive chirality for $SO(1,9)$, and thus obeys $\Gamma_{012\ldots9}\varepsilon = \varepsilon$, the condition is

$$B_2\varepsilon = -\varepsilon, \tag{4.50}$$

where (as in (2.7)) $B_2 = \Gamma_{3789}$. Likewise, the condition $\mu \in W_{\mathbf{4}'}$ corresponds to

$$B_2\varepsilon = \varepsilon. \tag{4.51}$$

As we know by now, many different boundary conditions preserve the supersymmetry of (4.50) or (4.51). As explained in Sect. 2.1.1, a particularly simple example arises for a system of D3-branes ending on an NS5-brane (or NS5-antibrane; the two choices correspond to the two possible conditions $B_2\varepsilon = \pm\varepsilon$). This corresponds to Neumann boundary conditions for gauge fields and for the scalar fields $\vec{X}$, extended to the fermions in a supersymmetric fashion.

Now let us describe what a generic choice of U would look like. We write $\mu = \eta \oplus \zeta$, $\eta \in W_{\mathbf{4}}$, $\zeta \in W_{\mathbf{4}'}$. Thus

$$\Gamma'\eta = \eta, \qquad \Gamma'\zeta = -\zeta. \tag{4.52}$$

For a suitable choice of basis, the inner product of μ with $\tilde{\mu} = \tilde{\eta} \oplus \tilde{\zeta}$ is

$$\langle \mu, \tilde{\mu} \rangle = \sum_{a=1}^{4} (\eta^a \tilde{\zeta}_a - \zeta_a \tilde{\eta}^a). \tag{4.53}$$

Since this inner product on W_8 is antisymmetric, a convenient way to proceed is to think of μ, η, and ζ as fermionic variables, and then the inner product can be described via a quadratic function of μ:

$$F(\mu) = \sum_{a=1}^{4} \eta^a \zeta_a. \tag{4.54}$$

A subspace $U \subset W_8$ is null if $F(\mu) = 0$ for $\mu \in U$. The simplification here is that there is no need to mention a second spinor $\tilde{\mu}$.

In this formulation, it is straightforward to describe the generic four-dimensional null subspace U. A generic four-dimensional subspace of W_8 can be defined by a condition

$$\zeta_a = \sum_b f_{ab} \eta^b \tag{4.55}$$

for some tensor f_{ab}. In order for this equation to imply that $0 = F(\mu) = \sum_a \eta^a \zeta_a$, the condition we need is that f should be symmetric, $f_{ab} = f_{ba}$. Here f transforms as the symmetric product $\mathbf{4}' \otimes \mathbf{4}'$, that is, like a selfdual three-form q (with constant coefficients) on $\mathbb{R}^{1,5}$. For $q = \sum_{I<J<K} q_{IJK} \mathrm{d}x^I \wedge \mathrm{d}x^J \wedge \mathrm{d}x^K$ (with I, J, K taking values in 012456), we can write (4.55) in terms of gamma matrices in the form[20]

$$\zeta = \sum_{I<J<K} q_{IJK} \Gamma^{IJK} \eta. \tag{4.56}$$

In terms of the supersymmetry generator ε, which we decompose as $\varepsilon = \varepsilon_+ + \varepsilon_-$ where $\Gamma' \varepsilon_\pm = \pm \varepsilon_\pm$, the condition is

$$\varepsilon_- = \sum_{I<J<K} q_{IJK} \Gamma^{IJK} \Gamma^3 \varepsilon_+. \tag{4.57}$$

This condition is familiar from (4.25), whose structure is hopefully now more clear.

Each choice of q gives a maximal null subspace U, but not every such subspace arises this way. The ones that so arise are precisely those that have trivial intersection with $W_{\mathbf{4}'}$, or in other words contain no vector with $\eta = 0$. Conversely, every maximal null subspace whose intersection with $W_{\mathbf{4}}$ is trivial can be defined by an equation

$$\eta^a = \sum_b g^{ab} \zeta_b, \tag{4.58}$$

where again g^{ab} is symmetric. Thus, g^{ab} transforms as an anti-selfdual three-form $\tilde{q}$ on $\mathbb{R}^{1,5}$. As in (4.56), we can equivalently write

$$\eta = \sum_{I<J<K} \tilde{q}_{IJK} \Gamma^{IJK} \zeta. \tag{4.59}$$

For NS5-brane boundary conditions, q vanishes; so for a small perturbation of those boundary conditions, q is small. When q is small, (4.56) is a good description. Close to the NS5-antibrane case, $\tilde{q}$ is small and (4.59) is more useful.

[20] If q is anti-selfdual, then as $\Gamma' \eta = \eta$, we have $q_{IJK} \Gamma^{IJK} \eta = 0$, giving another explanation for why in (4.56), q is selfdual.

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